Treating matrix nonlinearity in the binary model formulation for 3D ceramic composite structures

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Abstract
A computationally-efficient numerical approach to treating matrix nonlinearity in ceramic matrix composite components has been developed and validated. The model employs a dual mesh comprising strings of line elements that represent the fiber tows and 3D effective medium elements that define the external geometry and embody the matrix-dominated properties. Validation addressed test data for unnotched and open-hole tension specimens. For these tests, the onset of nonlinearity and subsequent plasticity due to matrix microcracking and interfacial debonding and sliding are satisfactorily represented by a linear Drucker–Prager model for failure initiation in the effective medium along with a fully-associated flow rule with isotropic, perfectly-plastic flow. Composite failure is assumed to be correlated with the maximum local stress averaged over a gauge volume dictated by the fiber tow width. Using one set of specimens for calibration, very good predictions of the nonlinear stress–strain response and fracture are obtained.

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1. Introduction
The insertion of ceramic matrix composites (CMCs) into hot structures is underway, but the rate of implementation has been slowed by the susceptibility of CMCs with 2D architectures to delamination. To overcome this limitation, multi-directional fiber architectures have been pursued, exploiting weaving and braiding technologies employed in the manufacture of polymer matrix composites (PMCs). Such integral textile CMC structures offer unprecedented breakthroughs in turbine engines, rockets, hypersonic flowpaths, and thermal protection systems (TPS) [9]. Integral textile CMCs have already demonstrated the following attributes: (i) robust thin skins to meet extreme heat transfer challenges without delamination; (ii) optimal deployment of fibers, including through-thickness, to match multi-axial stress fields; (iii) integral forming of transpiration holes, cooling slots, and large apertures, without cutting fibers; and (iv) integral attachment to other structures, such as active trusses for shape morphing. These features have been demonstrated for several types of CMCs, including C/SiC (in the context of rocket nozzles) as well as SiC/SiC and all-oxide composites (for turbine engine combustors and thermal protection systems).

The examples shown in Fig. 1 reveal the potential. They include: (i) an integrally woven panel with square channels for active cooling (Fig. 1a); (ii) pin-jointed attachment achieved by embedding an alloy truss structure into the fiber weave (Fig. 1b) and (iii) a candidate cooling feature for combustor liners comprising short parallel tubes incorporated at a surface step to launch a film of cooling air (Fig. 1c). The illustrated connection schemes provide compliant coupling between a hot CMC structure and a cool supporting structure, greatly reducing thermal mismatch stresses [7,9].

Notwithstanding this potential, the practical use of 3D CMC technology awaits a nonlinear code for optimizing design and predicting life subject to complex thermo-mechanical and environmental duty cycles. The absence of such a code has become a major barrier to the insertion of the technology into engines and other systems. One specific challenge in the case of CMCs is that, in many applications, structural robustness depends on the ability of the CMC composite to survive multiple cracking of the brittle matrix. The matrix cracking and consequent debonding of the fiber/matrix interfaces leads to substantial nonlinearity of the composite prior to failure. Not only must the nonlinearity be considered as a characteristic of the global structural performance of the composite, but its presence in regions of localized strain concentration is also vital to the toughness of the structure. Therefore, a model is needed that simultaneously accounts for the strong...
influence of the 3D fiber architecture and matrix nonlinearity on stress distributions and failure.

The principal objective of the present study was to devise, implement and assess a computationally-efficient, mechanics-based code to analyze nonlinear 3D CMC components. The computational formulation is an embellishment of the binary model of textile composites [3,5,6,10,12,14,15] with the new feature of nonlinear constitutive laws to represent the effects of progressive damage. The model employs a dual mesh comprising: (i) strings of 1D (line) elements representing the fiber tows, in one-to-one correspondence with the 3D interlacing pattern in the component; and (ii) solid elements, termed effective medium elements, that define the external component geometry and represent the matrix-dominated constitutive properties. The latter properties include the shear stiffness of local assemblies of fibers and matrix, the stiffness of such assemblies transverse to the fibers, and the stiffness of the matrix in the fiber direction. Thus the effective medium properties are closely related to those of the matrix alone, but are not identical to them. The dual mesh scheme is capable of dealing with complex, non-periodic architectures and asymmetric boundary conditions.

The dual mesh is combined automatically within a commercial finite element code (ABAQUS) by means of the multi-point constraint method. The dual mesh method facilitates convenient set-up by obviating the spatial matching of matrix and tow elements. The binary model formulation permits progressive refinement, so that the reduced-order representation of individual fiber tows used in the present work can be replaced by multi-element representations [5,13]. The variable-order formulation ensures numerical efficiency. Changing the order systematically shows that mesh-independent local strain variations are obtained provided strains are averaged over a gauge length comparable to the width of a single tow [13]. Importantly, experimental observations show that the same length scale embraces the physical processes that govern the important tensile, compressive, and microcracking damage mechanisms in a textile composite [6,16].

The present study focuses mainly on the inelastic deformation due to matrix microcracking and the associated debonding and sliding of fiber/matrix interfaces. Constitutive laws were calibrated using unnotched uniaxial tension tests and the model then validated by comparison with data for open-hole tensile tests. All tests were conducted using a SiC/SiC CMC with reinforcing fibers in an unbalanced 3D interlock woven structure (Fig. 2), with the load axis aligned with either the warp or weft tows. The SiC fibers (Sylramic, COIC Ceramics) were in 1600 denier tows, each consisting of 800 filaments of 10 μm diameter. Coatings of BN and SiC were deposited (by chemical vapor infiltration) on the fibers before forming the SiC matrix by a melt-infiltration process [2].

2. Formulation

2.1. Mesh generation

In the initial step, the external geometry of the system to be analyzed is defined by the effective medium mesh, set-up using a standard solid element mesh generator. The geometry of the fiber mesh is defined by a FORTRAN program that generates the nodal coordinates and the element connectivity. In the present calculations, each tow is represented by a string of 1D elements. Images of cross-sections show wavy tow loci (Fig. 2). These loci were idealized by piecewise linear tow element strings with dimensions indicated in Fig. 2. For warp tows, the strings constitute a trapezoidal wave (Fig. 2a); and for weft tows, a wave that alternates between trapezoidal and triangular (Fig. 2b). The tow mesh generation is facilitated by exploiting a unit cell that represents the periodic nature of the preform, although periodicity is not subsequently assumed in stress analysis. An open hole, if present, is created in the tow mesh by deleting tow elements. Where a tow is intersected by an internal or external specimen surface, a tow node that terminates the tow is automatically placed exactly on the surface. The FORTRAN program allows all of the salient parameters to be readily varied, including the width, length and thickness.
of the composite and the number, orientation, loci, and spacing of the tows.

In the second step, the nodal coordinates and element definitions for both the effective medium and the tows are incorporated into an ABAQUS input file. This file also assigns the properties of the effective medium and the tows, the cross-sectional area of the fibers (via the tow element stiffness) and the boundary conditions.

2.2. Constitutive laws

The constitutive properties needed to complete the model consist of the axial Young's modulus and strength of the fibers; Young's modulus and Poisson's ratio for the effective medium in the elastic regime (prior to matrix cracking); and a law to describe the behavior of the effective medium after cracking. Of these, the fiber modulus is specified a priori using manufacturer's data, while the fiber strength and both the elasticity and post-cracking yielding of the effective medium are calibrated against test data.

This modeling strategy departs significantly from that appropriate for polymer matrix composites, which have been much more studied in the computational mechanics literature. The reason is that significant differences arise in the level of knowledge available a priori for ceramic and polymer composites, especially regarding the character and properties of the matrix. Polymer composites are formed by very mature engineering processes, which yield nearly perfectly dense (porosity-free) and homogeneous matrices. Once the composite is formed, the elasticity of the matrix can be deduced from the composite elasticity by applying well-established micromechanical models. The inferred matrix elasticity is generally not noticeably different from the elasticity of neat specimens of the polymer cured in the same cycle. Therefore, in modeling polymer matrix textile composites, it is realistic to regard the elasticity of both the fibers and the matrix as known a priori;
accurate predictions of the composite elasticity have been shown to result [4,12].

In a ceramic matrix composite, in contrast, the matrix is formed by processes that commonly create a product that has substantial porosity and a microstructure that is not the same as that found in monolithic specimens of nominally the same material. The matrix material is peculiar to the geometrical environment (the interstices of fiber bundles) and the infiltration method (e.g., chemical vapor infiltration (CVI), reactive melt infiltration, or polymer precursor) that define the conditions of its formation. Unsurprisingly, the elasticity of a ceramic matrix can be very different to that of nominally equivalent monolithic specimens. In one study, for example, SiC matrices formed around carbon fibers by a combination of CVI and polymer precursor reaction were deduced by analysis of composite elasticity to be more than an order of magnitude more compliant than monolithic SiC [15]. In the present materials, the distribution of matrix within and between the fiber tows is known to be non-uniform, coating the outside of the specimen and the periphery of the tows more densely than other regions; and the matrix is known to contain many initial microcracks and small pores. Therefore, the elasticity of the matrix cannot be considered known a priori. Instead, it must be inferred (via calibration of the effective medium modulus) by a test on a composite specimen. Recognizing the uncertainty of the matrix modulus is especially important for ceramic composites, where the matrix stiffness is commonly comparable to the fiber stiffness; indeed the matrix material in monolithic form is often stiffer than the fibers, but more compliant than the fibers in the form it takes within the composite.

In both polymer and ceramic matrix composites, the nonlinear properties of the matrix that prevail after matrix damage has begun (crazing, microcracking or other damage, as the case may be) also cannot be predicted a priori and must be calibrated. Again the calibration must be performed on a composite specimen, because, like the character of the ceramic matrix, damage mechanisms are governed by the geometrical environment of the fiber tows and are dissimilar to those that arise in monolithic matrix specimens. This is the case in both polymer [11] and ceramic composites.

2.2.1. Fiber tows

Prior to fiber fracture, the axial response of each fiber tow element is characterized by the spring stiffness $k_t = A_tE_t$, where $A_t$ is the cross-sectional area of fibers within a single tow and $E_t$ is the Young’s modulus of the fibers (see Appendix A for notes on this modeling choice). Since the tow elements are one-dimensional, only the spring stiffness can be carried forward into the calculation, even though ABAQUS requires the fiber modulus and fiber cross-sectional area to be input separately.

The strength, $\sigma_0$, of the tow segments can be taken to be either deterministic or stochastic. If the strength is stochastic, the cumulative effects of filament fractures and subsequent frictional sliding within each tow can be represented by a 1D nonlinear constitutive law obtained from a shear lag analysis of a unidirectionally reinforced CMC undergoing stochastic fiber fragmentation [8]. However, analyses conducted using the simpler choice of elastic fibers with deterministic strength in conjunction with nonlinear effective medium behavior replicate the important inelastic effects for the composite in the tests considered in the present study, as demonstrated below. Preliminary calculations were also performed to examine the other options. Incorporating discrete tow fractures, as required to simulate stochastic tow fracture, proved problematic (and unnecessary in the present work) because the ensuing stress waves caused significant numerical ringing and instability. Furthermore, the inelastic strains due to fiber fragmentation only become appreciable as the ultimate strength is approached and have minimal effect on the stress redistribution around holes.

2.2.2. Effective medium elements

The inelastic strains due to matrix microcracking and interfacial debonding/sliding are assigned to the effective medium elements. In the absence of direct measurements of the nonlinear properties of the matrix, a parametric constitutive law must be chosen for the effective medium, with a functional form that is consistent with the micromechanics of matrix failure and can account for observed composite behavior. An appropriate choice for analyzing the aligned tests studied in this paper is a simple elastic–perfectly plastic constitutive law, with yield beginning at the onset of matrix cracking and with a fully-associated flow rule (i.e., plastic strain increments normal to the yield surface). In this scheme, nonlinear hardening in the stress–strain response of the composite arises solely from coupling and stress transfer between the effective medium and fiber tow elements.

For convenient implementation in the ABAQUS code, the yield criterion for the effective medium elements is taken to be defined by a linear form of the Drucker–Prager (DP) criterion, a pressure-dependent plasticity law commonly used for geological materials. This law can be written in terms of the local principal stresses, $\sigma_1$, $\sigma_2$, $\sigma_3$, as:

$$\sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{6}} = A + B(\sigma_1 + \sigma_2 + \sigma_3) \quad (1)$$

The two material constants $A$ and $B$ are set by assuming, on micro-mechanical grounds, that a critical tensile stress criterion governs two loading scenarios: (i) uniaxial tension ($\sigma_1 = \sigma_c$, $\sigma_2 = \sigma_3 = 0$), where $\sigma_c$ is the matrix cracking stress; and (ii) hydrostatic tension ($\sigma_1 = \sigma_2 = \sigma_3 = \sigma_c$). Substituting these two conditions into Eq. (1) yields:

$$A = \frac{\sqrt{3}}{2} \sigma_c, \quad B = -\frac{1}{2\sqrt{3}}$$

With these constants set in this manner, the DP criterion provides a smooth yield surface that, in the tensile region, falls mostly between the surfaces of maximum principal strain and maximum principal stress criteria for matrix cracking; and is quite close to

![Fig. 3. Schematic of the yield surface defined by the linear version of the Drucker–Prager model. Also shown for comparison are the yield surfaces for the von Mises criterion as well as those based on maximum principal stress and maximum principal strain.](image-url)
the maximum principal strain surface in the region of tensile biaxial stress (Fig. 3).

The nonlinear response of the effective medium elements is thus defined with a single parameter, \( \sigma_c \), which must be calibrated with experimental data.

3. Model calibration

The fiber strength \( \sigma_t \) was calibrated by comparing the predicted and measured composite ultimate tensile strengths for unnotched test samples aligned with the load axis parallel to the warp or weft fiber directions. The fiber strength was defined as the product of the measured composite strain at failure and the fiber modulus.

The elastic modulus \( E_{em} \) and cracking stress \( \sigma_c \) of the effective medium elements were obtained through an iterative best-fit process involving numerical simulations of the unnotched tensile stress/strain curves and comparison with the corresponding experimental measurements. The metrics used for assessment of the quality of fit included the composite Young’s modulus, the stress at the onset of nonlinearity and the tangent modulus in the inelastic regime.

The constitutive properties inferred from this calibration procedure, applied separately to two sets of data from tests with loading in the warp and weft directions, are summarized in Table 1. In both cases, the calibrated model was in excellent agreement with the calibration data over the entire strain range (Fig. 4). However, the inferred property values for the effective medium were significantly different for the two sets of data. Such anisotropy is expected in an unbalanced weave. For instance, the value of \( E_{em} \) inferred from the tensile data in the warp direction is influenced by the fiber content in the weft direction, and vice versa. Similar effects arise in the inferred values of \( \sigma_c \). The implications are addressed below.

4. Influence of textile architecture

The potential influence of the waviness of the warp and weft fiber tows on the macroscopic composite stress–strain response was assessed by comparing calculations for the correct tow architecture (Fig. 2) with calculations in which the tows in the loading direction were straight. One such example, for loading in the weft direction, is shown in Fig. 4. The straight-tow calculation yields significantly higher composite stiffness, flow stress, and tangent modulus after yield. Therefore, the tow waviness causes a significant knockdown in these properties.

Additional insights are gleaned from inspection of the spatial distributions in the stresses and strains as well as the variations in these quantities from their mean values. For instance, Fig. 5 shows that the tow stresses in the weft direction vary by as much as \( \pm 20\% \) of their mean value at a prescribed applied strain. One implication is that failure prediction based on average tow stresses may yield non-conservative results. The origins of these variations are twofold. First, the tow undulations lead to periodic stress variations along the length of each individual tow (Fig. 6a and b). A second (more subtle) effect arises near the specimen edges. Because of the reduced constraint at the edges, the effective medium elements exhibit strain elevations in these regions (Fig. 6c and d). In turn, the constraint they exert in retarding tow straightening is reduced, thereby lowering the tow stresses, by 15–25% (Fig. 6a and b).

Spatial strain variations similar to those seen in Fig. 6 have been observed previously in binary model calculations for both ceramic and polymer matrix textile composites [15,16]. However, those calculations were confined to the elastic regime and therefore the variations in the effective medium strains were relatively small and stress relief in near-edge tows was negligible.

5. Validation and implementation

The predictive capability of the calibrated model was tested by comparing calculated stress–strain relations with experimental data for open-hole tension tests in both the warp and weft orientations. Table 1

<table>
<thead>
<tr>
<th>3D angle interlock</th>
<th>Warp</th>
<th>Weft</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiber volume fraction (%)(^a)</td>
<td>9</td>
<td>19</td>
</tr>
<tr>
<td>Fiber modulus, ( E_f ) (GPa)(^a)</td>
<td>395</td>
<td>395</td>
</tr>
<tr>
<td>Effective medium modulus, ( E_{em} ) (GPa)(^b)</td>
<td>74</td>
<td>103</td>
</tr>
<tr>
<td>Effective medium Poisson’s ratio, ( \nu ) (^c)</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>Effective medium cracking stress, ( \sigma_0 ) (MPa)(^b)</td>
<td>48</td>
<td>83</td>
</tr>
</tbody>
</table>

\(^a\) Prescribed properties. \(^b\) Properties obtained from calibrations.
tations. Typical computed stress and strain distributions in the tow elements and effective medium elements are shown in Fig. 7. Tensile stress and strain concentrations are evident adjacent to the hole, while stresses and strains are relieved upstream and downstream of the hole.

For simulations such as Fig. 7 involving loads applied in the warp or weft directions, a simplified approximate method was used to account for the anisotropy observed in the calibrated effective medium properties (Table 1). For open-hole tests in the weft orientation, the effective medium was regarded as isotropic with properties inferred from the unnotched tests in the weft orientation (right column in Table 1). Similarly, for tests in the warp orientation, the effective medium was again taken to be isotropic, but with reduced property values, obtained from the corresponding unnotched tests. This isotropic effective medium approximation is expected to be adequate for such aligned tests, since the transverse composite strain is dominated by the orthogonal fiber

Fig. 6. Fiber stresses (top) and axial strains in effective medium (bottom), at applied composite strain of \( \varepsilon_C \), where \( \varepsilon_C \) is the cracking strain.

Fig. 7. Distributions in (a) tow stresses and (b) effective medium strains in an open-hole tensile specimen loaded in the warp direction at an applied stress of 130 MPa. (For clarity, weft tows are not shown in (a)).

Fig. 8. Measurements and predictions of local strains in open-hole tension specimens, with loading in weft and warp directions. The hole diameter is 9.4 mm and the specimen width is 25.4 mm. Open circles denote predicted rupture points. Strains were measured with 3 mm \( \times \) 3 mm strain gauges mounted on the broad face of the specimen adjacent to the hole along the incipient fracture plane (shown in inset). Accordingly, the strains from the simulations were averaged over the same 3 mm \( \times \) 3 mm area.
set and is the only component of strain where anisotropy in the effective medium would come into play. For predictions at other orientations, anisotropy in the elastic properties and yield criterion may need to be taken into account.

Comparison of the predictions and measurements for the open-hole tension tests is based on: (i) the strain evolution in regions adjacent to the hole, where failure is initiated; and (ii) the ultimate net-section tensile strength. The computed strains used for validation were averaged over a square area that matched the 3 mm × 3 mm areas covered by the strain gauges (see inset in Fig. 8). The edge length of this area is comparable to the unit cell of the weave structure. The inelastic strains calculated in this manner (Fig. 8) agree well with the measurements, except for the underestimate close to the ultimate strength. This deviation is likely caused by fiber fragmentation in the vicinity of the strain gauges.

Predictions of ultimate strength were made by assuming that rupture occurs when the average axial stress in the 3 mm × 3 mm area adjacent to the hole reaches the unnotched tensile strength (Fig. 4). The predicted rupture stresses (shown as open circles in Fig. 8) are virtually identical to the measured values for both orientations. Evidently, the additional inelastic strain associated with fiber fragmentation has minimal effect on the ultimate strength.

6. Conclusions

A computationally-efficient finite element method for analyzing textile composites was extended to treat the nonlinear mechanical behavior of 3D CMCs. For unnotched and open-hole tension specimens in which the load was aligned with either warp or weft fiber tows, the onset of nonlinearity and subsequent plasticity due to matrix microcracking and interfacial debonding and sliding are satisfactorily represented by a single-parameter criterion for failure initiation in the effective medium along with a fully-associated flow rule with isotropic, perfectly-plastic flow. Failure of the composite material is assumed to be correlated with the maximum value of the local stress averaged over a gauge volume dictated by the width of a single tow. Using one set of unnotched composite specimens to calibrate the effective medium behavior and fiber bundle failure, very good predictions of the nonlinear stress-strain response and ultimate strength of open-hole specimens are obtained. Gauge-averaged stresses are mesh-independent, in spite of the simplicity of the meshing scheme in the binary model.

The importance of representing the textile architecture is emphasized by the strong variations in the strain fields predicted for both the effective medium and tow elements on gauge lengths comparable to the tow width. As well as stress concentrations around an open hole, some relaxation of the stresses in edge tows is predicted due to the large matrix deformation that occurs preferentially where constraint is lowered. Given the spatial variations in strains, predictions of failure using a homogenized material model would be implausible.

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Appendix A. The prescription of the fiber tow modulus

In prior work applying the binary model to polymer composites, the fiber tow element stiffness was defined by

$$k_t = A_t/E_m$$

(A1)

where $$A_t$$ is the cross-sectional area of the fibers in the tow [5,12,16]. Subtracting the term $$E_m$$ in Eq. (A1) avoids double counting of the stiffness contribution of the effective medium in the volume that is occupied by fibers in the real composite but is not occupied by fibers in the model, because the tow elements are one-dimensional (zero volume). The subtraction is necessary when the fiber and matrix modulus are being specified a priori; without it, incorrect predictions result for the composite modulus when the model is applied to an ideal unidirectional composite.

A definition similar to Eq. (A1) could be employed in the present work. But, if it were, a modification would be necessary in the regime where the effective medium is nonlinear: the spring constant $$k_t$$ would have to be defined as the tangent stiffness of the fiber tow and the stiffness subtracted out would have to be the tangent stiffness of the effective medium. In the elastic regime, the result would be the same as Eq. (A1), but in the nonlinear regime, where the tangent stiffness of the effective medium is zero, Eq. (A1) would simplify to $$k_t = A_tE_m$$. Only with this modification would correct moduli be predicted for an ideal unidirectional composite in both the linear and nonlinear regimes. However, because the modulus of the ceramic matrix in the present work is not known a priori but must be calibrated, a simpler definition of the fiber modulus than described in the preceding paragraph will also suffice, which is to assign $$k_t = A_tE_m$$ in both the linear and nonlinear regimes, with no subtraction of the effective medium modulus. Calibration of $$E_m$$ will assure the correct composite modulus in the elastic regime, while the composite tangent modulus in the nonlinear regime will also be correct. The only detriment to this simpler assignment is that, all other things being equal, different calibrated values of $$E_m$$ in the elastic regime might be expected for composites with different fiber volume fractions. However, a ceramic matrix is likely to be different when formed between different fiber populations anyway: the effective medium calibration must be repeated for each significantly different value of fiber volume fraction. This being the case, the simple tow element stiffness assignment used in this paper is preferable.

Appendix B. Formula for the plastic strain increment

Plastic flow is defined by a fully-associated flow rule. That is, the plastic potential is taken to be identical to the yield function and thus the incremental plastic strain is given by $$\delta \varepsilon_p = d\delta F/\partial \sigma_i$$, where $$\delta \varepsilon_p$$ is a non-negative plastic multiplier. If the components of the strain increment are mapped onto stress space by a scalar multiplier, $$d\delta F = \alpha \cdot d\delta \sigma_i$$ for arbitrary $$\alpha$$, the strain increment maps onto a vector that is normal to the yield surface. The plastic multiplier $$\delta \varepsilon_p$$ is determined by the consistency condition

$$dF(\sigma_i) = (\partial F/\partial \sigma_i)\delta \sigma_i = 0$$

(B1)

which ensures that the updated stress state $$\sigma_i + \delta \sigma_i$$ also satisfies the yield condition, $$F(\sigma_i + \delta \sigma_i) = 0$$. An explicit expression of the incremental stress–strain relation can then be derived as follows [1].

For elastic–perfectly plastic materials, any stress increment is supported entirely by the elastic strain increment, i.e.,

$$\delta \varepsilon_i = C_{ijkl}d\sigma_{ij} = C_{ijkl}d\delta \sigma_i - C_{ijkl}d\delta \varepsilon_i$$

(B2)

where $$d\sigma_{ij}$$ and $$d\delta \sigma_i$$ are the total and plastic incremental strains, respectively, and $$C_{ijkl}$$ is the stiffness tensor. Combining this result with the consistency condition (Eq. (B1)) gives
\[ d\lambda = \frac{(\partial F/\partial \sigma_{ij}) C_{ijkl}}{(\partial F/\partial \sigma_{pq}) C_{pqrs}(\partial F/\partial \sigma_{rs})} d\epsilon_{kl} \]  

(B3)

The incremental stress–strain relation can then be expressed explicitly as

\[ d\sigma_{ij} = \frac{C_{ijkl}}{C_{ijmn} (\partial F/\partial \sigma_{mn}) C_{pqrs} (\partial F/\partial \sigma_{rs})} d\epsilon_{kl} \]  

(B4)

When the current stress state \( \sigma_{ij} \) is known and the incremental strain \( d\epsilon_{ij} \) is prescribed, the corresponding stress increment \( d\sigma_{ij} \) is uniquely determined by Eq. (B4).

References