

Alternating nonzero automata

Application to the satisfiability of $\text{CTL}^*[\exists, \forall, \mathbb{P}_{>0}, \mathbb{P}_{=1}]$

Hugo Gimbert, joint work with Paulin Fournier

LaBRI, Université de Bordeaux
ANR Stoch-MC

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Control and verification of probabilistic systems

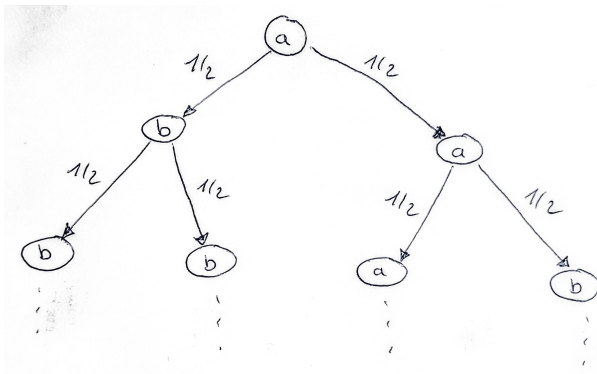
- ▶ Logic (specification): $\text{CTL}^*[\exists, \forall, \mathbb{P}_{>0}, \mathbb{P}_{=1}]$
- ▶ Automata (compilation): nonzero automata.
- ▶ Games (algorithms): global winning condition.

Binary trees

Full binary tree $\{0, 1\}^*$ labelled with $\{a, b\}$

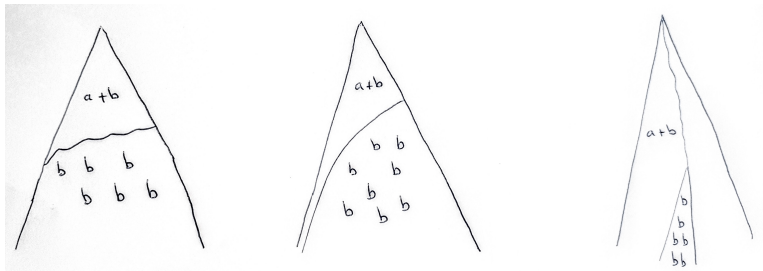
Random walk, Markov chain

A node = flip fair coin



Trees whose branches have finitely many a

- ▶ for sure
- ▶ almost-surely
- ▶ with positive probability



Nonzero automata [Bojanczyk,16]

$$A = (Q, \geq, \Sigma, \Delta, F_{\forall}, F_1, F_{>0})$$

- ▶ (Q, \geq) ordered finite set of states
- ▶ Alphabet Σ
- ▶ Transitions $\Delta \subseteq Q \times \Sigma \times Q \times Q$
- ▶ Acceptance conditions $F_1 \subseteq F_{\forall} \subseteq Q$ and $F_{>0} \subseteq Q$

Run on input tree $t : \{0, 1\}^* \rightarrow \Sigma$

$\rho : \{0, 1\}^* \rightarrow Q$ consistent with Δ

Acceptance condition

- ▶ Limsup of a branch Q_{∞}
- ▶ $Q_{\infty} \in F_{\forall}$ for every branch
- ▶ $Q_{\infty} \in F_1$ for almost every branch
- ▶ Each time the run enters $F_{>0}$ it stays in $F_{>0}$ with positive probability

Example

- ▶ Below every a there is a b
- ▶ Below every a there is positive probability to never see b
- ▶ Almost surely a branch has finitely many b

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△ $(q, a, (?, A)), (q, a, (A, ?)), (q, b, (B, B))$

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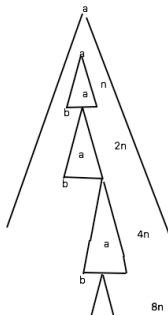
(does not see B infinitely often)

$F_{>0} \ ?, A$

positive probability to never see b again

Example

- ▶ Below every a there is a b
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- ▶ Almost surely a branch has finitely many b



Deciding Emptiness

Theorem (Bojanczyk, Fournier, Gimbert, Kelmendi, 17)

Emptiness of nonzero automata is decidable in $\text{NP} \cap \text{coNP}$.

- ▶ PTIME for F_{\forall} -trivial automata ($F_{\forall} = Q$),
- ▶ Equivalent to solving a parity game of exponential size.
- ▶ Strategies of the players can be represented concisely.

Alternating nonzero automata

$$A = (Q, Q_E, Q_A, \leq, \Sigma, \Delta, F_{\forall}, F_1, F_{>0})$$

- ▶ Two players Eve and Adam
- ▶ Q partitioned in Q_E, Q_A
- ▶ Local transitions $(q, a, r) \in \Delta$ (stays in place)
- ▶ Split transitions $(q, a, q_0, q_1) \in \Delta$ (moves in the tree)

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Acceptance game on input tree $t : \{0, 1\}^* \rightarrow \Sigma$

- ▶ **Vertices** = pairs (n, q) with n node in $\{0, 1\}^*$ and $q \in Q$,
- ▶ **Local move**: $(n, q) \rightarrow (n, r)$ whenever $(q, t(n), r) \in \Delta$,
- ▶ **Split move**: $(n, q) \rightarrow \frac{1}{2}(n0, q_0) + \frac{1}{2}(n1, q_1)$ whenever $(q, t(n), q_0, q_1) \in \Delta$

Eve wins if $Q_{\infty} \in F_{\forall}$ for sure, $Q_{\infty} \in F_1$ almost-surely and whenever the play enters $F_{>0}$ it stays in $F_{>0}$ with positive probability.

Alternating nonzero automata

Closure properties

- ▶ Intersection (Adam choice)
- ▶ Union (Eve choice)
- ▶ Complement ? open
- ▶ Emptiness? open

A nice sub-class

- ▶ Similar to limited-alternation automata (Kupferman Vardi Wolper)
- ▶ One canonical choice for Adam in each state, other choices strictly increase the rank.
- ▶ \implies ultimately only Eve has control

Limited choice

Lemma

The acceptance game is determined: either Eve or Adam has a winning strategy. Moreover, positional strategies are enough for Eve to win the acceptance game.

A positional strategy of Eve $\sigma : \{0, 1\}^* \times Q_E \rightarrow \Delta$ represented as a tree $\{0, 1\}^* \rightarrow \Delta^{Q_E}$.

Theorem

The language of positional winning strategies of Eve can be recognized by a non-deterministic automaton of exponential size.

Corollary: emptiness is decidable in $\text{NEXPTIME} \cap \text{co-NEXPTIME}$.

Theorem

The language of positional winning strategies of Eve can be recognized by a non-deterministic automaton of exponential size.

- ▶ **Deterministic automata:** guess a play with $Q_\infty \notin F_\forall$
- ▶ **Nonzero automata:** check that all probability distributions that Adam can generate are surely+almost-surely+positively accepting.
- ▶ **Technical challenge:** checking in parallel several nonzero-conditions.

Satisfiability of CTL* $[\exists, \forall, \mathbb{P}_{>0}, \mathbb{P}_{=1}]$

State formulas ϕ . Path formulas ψ .

$$\phi = a \in \Sigma \mid \forall\psi \mid \exists\psi \mid \mathbb{P}_{>0}(\psi) \mid \mathbb{P}_{=1}(\psi)$$

$$\psi = X\psi \mid \psi U\psi \mid G\psi \mid \phi$$

+ boolean connectives.

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Conversion from $CTL^*[\exists, \forall, \mathbb{P}_{>0}, \mathbb{P}_{=1}]$ to alternating automata.

	$[\exists, \forall, \mathbb{P}_{>0}, \mathbb{P}_{=1}]$
CTL^*	$3\text{-NEXPTIME} \cap \text{co-}3\text{-NEXPTIME}$
CTL	$\text{NEXPTIME} \cap \text{co-NEXPTIME}$

Known results [Brazdil et al.].

	$[\mathbb{P}_{>0}, \mathbb{P}_{=1}]$
CTL^*	3-EXPTIME
CTL	EXPTIME

Conclusion

- ▶ Alternating nonzero automata with limited choice.
- ▶ Application: satisfiability of $\text{CTL}^*[\exists, \forall, \mathbb{P}_{>0}, \mathbb{P}_{=1}]$.

Future work

- ▶ Complementing and deciding emptiness in general.
- ▶ Games.
- ▶ Decidability of MSO?