

# Uniform control of a population of identical MDPs

Ongoing work – Preliminary report

Nathalie Bertrand

ANR STOCH-MC meeting - 4 March 2015

# Family of identical MDPs

Motivation: Modelisation of the reaction of a population of cells to which the same amount of drug is applied.

PFA natural model for an infinite population

What about handling large but finite populations?

# Family of identical MDPs

Motivation: Modelisation of the reaction of a population of cells to which the same amount of drug is applied.

PFA natural model for an infinite population

What about handling large but finite populations?

- ▶  $\mathcal{P}_n$ : population of  $n$  identical MDPs
- ▶ Controller: based on the whole configuration chooses an action that is applied uniformly to each MDP
- ▶ Goal: control  $\mathcal{P}_n$  so as to achieve a given global objective
- ▶ Global objective: the **maximum probability** that a **given proportion** of MDPs **stay in/reach a target set of states** **meets a threshold**.

# Parameterized verification

## **Fixed $n$**

one can build the product MDP for  $\mathcal{P}_n$ , identify good configurations and decide whether a controller exists.

# Parameterized verification

## Fixed $n$

one can build the product MDP for  $\mathcal{P}_n$ , identify good configurations and decide whether a controller exists.

## Parameter $n$

verify a global objective for all sizes  $n$

Global objective: the maximum probability that a given proportion of MDPs stay in/reach a target set of states meets a threshold.

$$\forall n \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_n \models \text{Always at least 80\% of MDPs in T}) \geq .7$$

three P's: property (Always/Eventually), proportion and probability

# Solving always proportion 100% with probability 1

Amounts to solving a safety question almost-surely on the individual MDP viewed as a PFA.

$$\forall n \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_n \models \text{Always all MDPs in } T) = 1$$



$$\max_w \mathbb{P}_w(\mathcal{P} \models \text{Always } T) = 1$$

**Rk:** in this case, the PFA can be continuously approximated by  $\mathcal{P}_n$  for increasing sizes  $n$ .

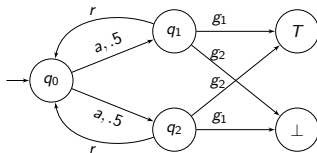
# Solving eventually proportion 100% with probability 1

The limit behaviour when  $n$  tends to infinity does not coincide with viewing the MDP as a PFA transforming distributions!

$$\forall n \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_n \models \text{Eventually all MDPs in } T) = 1$$



$$\max_w \mathbb{P}_w(\mathcal{P} \models \text{Eventually } T) = 1$$



- ▶ For every  $n$ , there exists a strategy that ensures eventually all MDPs in  $T$  with probability 1.
- ▶ In the PFA, the maximum probability to reach  $T$  is .5.

# Solving eventually proportion 100% with probability 1

$$\forall n \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_n \models \text{Eventually all MDPs in } T) = 1$$



$$\max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_{|Q|} \models \text{Eventually all MDPs in } T) = 1$$

for  $|Q|$  the number of states of the (individual) MDP



# Solving eventually proportion 100% with probability 1

$$\forall n \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_n \models \text{Eventually all MDPs in } T) = 1$$



$$\max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_{|Q|} \models \text{Eventually all MDPs in } T) = 1$$

for  $|Q|$  the number of states of the (individual) MDP

## Proof idea

- ▶ Monotonicity property:

$$\max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_{n+1} \models \text{Eventually all MDPs in } T) = 1$$

$$\Rightarrow \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_n \models \text{Eventually all MDPs in } T) = 1$$

- ▶ Cutoff:

$$\max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_{|Q|} \models \text{Eventually all MDPs in } T) = 1$$

$$\Rightarrow \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_{|Q|+1} \models \text{Eventually all MDPs in } T) = 1$$

# Solving eventually positive proportion with probability 1

$$\forall n \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_n \models \text{Eventually at least one MDP in } T) = 1$$



$$\max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_1 \models \text{Eventually the MDP in } T) = 1$$

# Solving eventually positive proportion with probability 1

$$\forall n \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_n \models \text{Eventually at least one MDP in } T) = 1$$



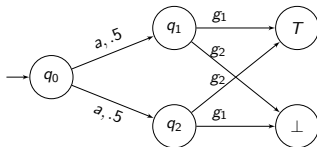
$$\max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_1 \models \text{Eventually the MDP in } T) = 1$$

## Proof idea

- ▶ Monotonicity property:  
 $\max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_n \models \text{Eventually at least one MDP in } T) = 1$   
 $\Rightarrow \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_{n+1} \models \text{Eventually at least one MDP in } T) = 1$
- ▶ Other direction by contradiction.

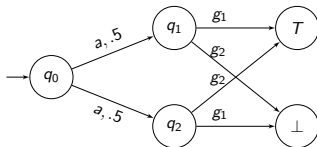
## Quantitative questions on examples

$\forall n \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_n \models \text{Eventually at least 50\% of MDPs in T}) = 1$

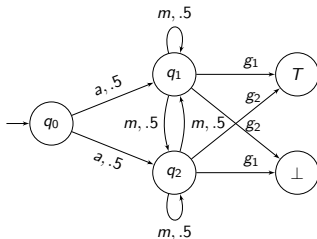


# Quantitative questions on examples

$$\forall n \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_n \models \text{Eventually at least 50\% of MDPs in T}) = 1$$



$$\forall p \forall n \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_n \models \text{Eventually at least p\% of MDPs in T}) = 1$$



# Future work

- ▶ Identify subcases for which the discrete approximation of PFA by family of MDP works.
- ▶ Move towards quantitative questions.

$$\forall n \max_{\sigma} \mathbb{P}_{\sigma}(\mathcal{P}_n \models \text{Always at least 80\% of MDPs in T}) \geq .7$$

- ▶ M1 intern from CMI this summer