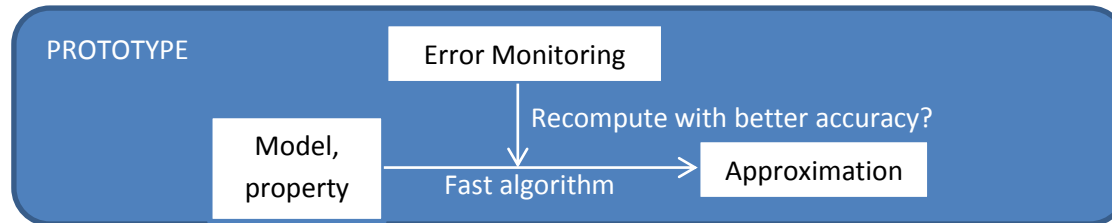


# Error bounds

# General Idea

Approximate the values.

Instead of e.g. Probability = 0.3, we answer e.g. Proba in [0.2,0.4].  
If question was: is probability < 0.5, then sufficient.



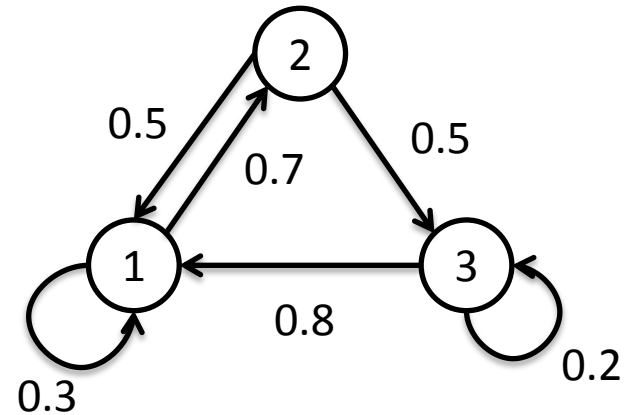
Multiple frameworks:

- **Bounded** Model Checking of big Dynamic Bayesian Networks
- Model Checking of Markov Chains vs distribution based logics
- Model Checking of MDP vs distributions ? [Chada et al.'11]

# Bounded model checking of DBN

# Irreducible aperiodic chains

$$M = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0.5 & 0 & 0.5 \\ 0.8 & 0 & 0.2 \end{pmatrix}$$

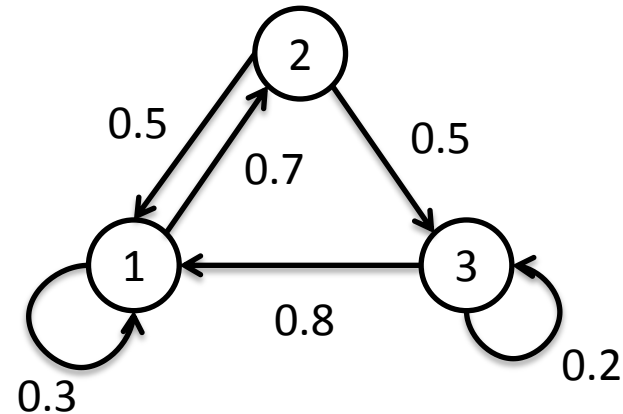


M is **irreducible aperiodic** because:

$$M^2 = \begin{pmatrix} .44 & .21 & .35 \\ .55 & .35 & 0.1 \\ .4 & .56 & .04 \end{pmatrix}$$

# Contracting Factor

$$M = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0.5 & 0 & 0.5 \\ 0.8 & 0 & 0.2 \end{pmatrix}$$



If  $M$  is **irreducible aperiodic**, then  $\|Mu - Mv\| \leq \alpha \|u - v\|$

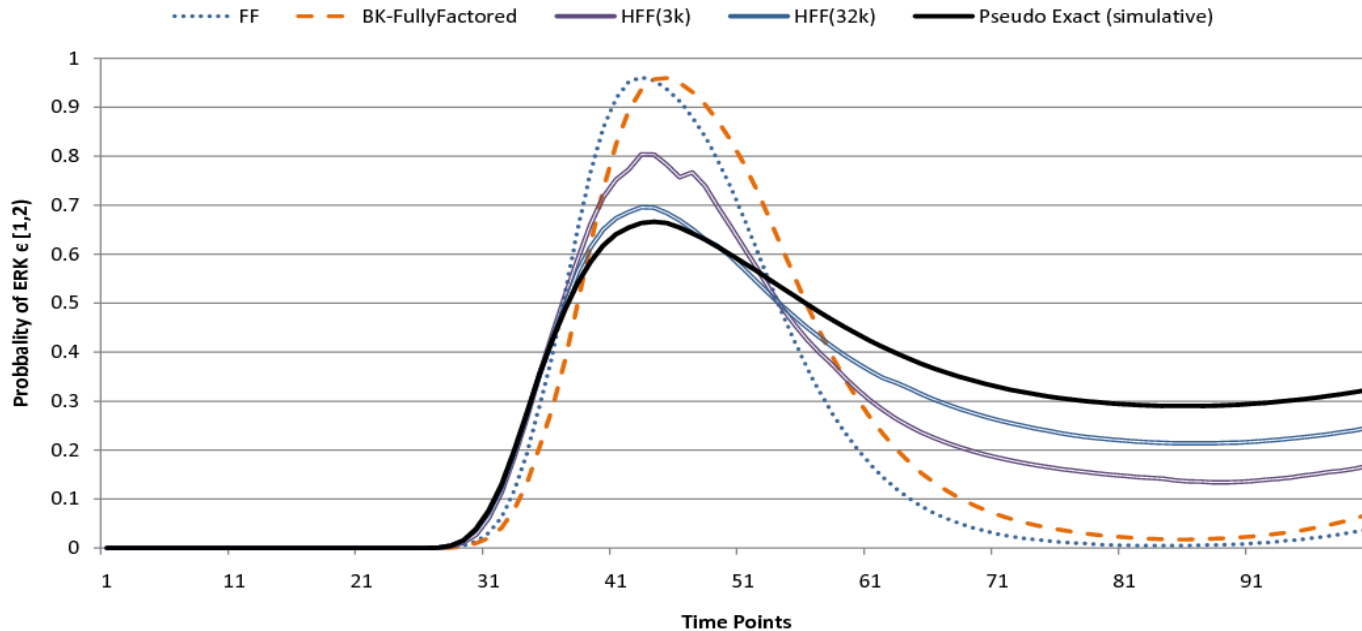
Where  $\alpha < 1$  is the **contracting factor**

and  $\|\cdot\|$  is L1 (for instance) norm.

$\longrightarrow$  An error made at time  $t$  for  $M^t \mathbf{u}$   $\longrightarrow$  Approx. For bounded  
 has limited influence over  $M^{t+t'} \mathbf{u}$   $\longrightarrow$  model checking

# What we already have:

- 1) Very Large Markov Chains encoded as **Dynamic Bayesian Networks** + statistics or approximated inference (**bounded paths**).



Parametrized algorithms to compute more and more accurate probabilities.  
Not accurate enough in reasonable time. No full error analysis.

# What we already have:

2) Computation of bound on error made in the approximated inference. => **some result will be certain.**  
**But proving Proba=1 not possible.**

Factored Frontier (*Murphy and Weiss*), BK (*Boyen and Koller*), Hybrid FF...

One step error,  
depend on algo

Error at time t

$$\Delta^t \leq \epsilon_0 \left( \sum_{j=0}^t \beta^j \right) \leq \epsilon_0 \left( \sum_{j=0}^{\infty} \beta^j \right) = \frac{\epsilon_0}{1 - \beta}$$

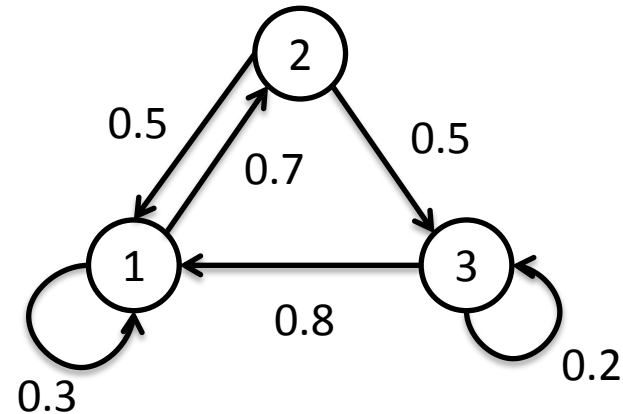
Factor depending on the  
underlying Markov Chain

model checking of MC  
against distribution based  
logics



# Contracting Factor

$$M = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0.5 & 0 & 0.5 \\ 0.8 & 0 & 0.2 \end{pmatrix}$$



If  $M$  is **irreducible aperiodic**, then

- unique stationary distribution  $\mathbf{f} = M\mathbf{f}$
- each trajectory converges towards  $\mathbf{f}$

→ « Know » the stationary behavior. → Model Check.  
Converge fast, in  $\alpha^t$ . → of irred aper?

# Model Checking of MC vs distribution logic

Set **Init** given by interval of probabilities over each state.

**Ex:**  $a[\text{sunny}] \in [0.7,1]$ ;  $a[\text{rainy}] \in [0,0.3]$ .

**Question:** For all  $a \in \text{Init}$ , «for all  $t$ ,  $M^t a[1] \geq 0.7$ »

**First Idea to solve question:**

1) Extract intervals:  $I=[0,0.3]$ ,  $J=(0.3,0.7)$ ,  $K=[0.7,1]$ .

2) Look at **symbolic trajectories**

**ex:**  $C^*$ ,  $DC^*$ ,  $DDC^*$  with  $C=(K,I)$  and  $D=(J,J)$

3) Build symbolic language  $L^{\text{Init}}_M \in (\{I,J,K\}^2)^*$  of  $M$ , that is

$D_1 \dots D_k \in L_M$  iff there exists  $a \in \text{Init}$  with  $M^i a[j] \in d_i[j]$

here:  $L^{\text{Init}}_M = \{C^*\}$

If  $L^{\text{Init}}_M$  is regular, then we can answer the question

# Model Checking of MC vs distribution logic

If  $L^{\wedge} \text{Init\_M}$  is regular, then we know how to proceed.

Result:  $L^{\wedge} \text{Init\_M}$  is not regular for some irreducible aperiodic MC with 3 states, even with a unique initial configuration ( $\Rightarrow$  symbolic trajectory not ultimately periodic)

[Agrawal, Akshay, G., Thiagarajan, JACM'14]

$$M = \begin{pmatrix} 0.6 & 0.1 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}$$

Init: (1/4, 1/4, 1/2)

Discretization  $I=[0, 1/3]$ ,  $J=(1/3, 1]$ .

(Question : Is  $M^t[1] > 1/3$  for some  $t$ ?)

# Model Checking of MC vs distribution logic

$$M = \begin{pmatrix} 0.6 & 0.1 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}$$

Init: (1/4,1/4,1/2)

Discretization I=[0,1/3], J=(1/3,1].

$\mathbf{u} = (1/3, 1/3, 1/3)$ ,  $\mathbf{v} = (-1 - \sqrt{3}i, -1 + \sqrt{3}i, 2)$ ,  $\mathbf{w} = (-1 + \sqrt{3}i, -1 - \sqrt{3}i, 2)$

$M \mathbf{u} = \mathbf{u}$

$M \mathbf{v} = \rho e^{i\theta} \mathbf{v}$

$M \mathbf{w} = \rho e^{-i\theta} \mathbf{w}$

Where  $\rho = \sqrt{19}/10$  and  $\theta = \cos^{-1}(4/\sqrt{19})$ .

$\mathbf{u}, \mathbf{v}, \mathbf{w}$  are 3  
eigen vectors:  
They form a basis

We decompose  $\text{Init} = (1/4, 1/4, 1/2) = \alpha \mathbf{u} + \beta \mathbf{v} + \gamma \mathbf{w}$ .

$M^n \text{init}[1] \in J$  iff  $(\alpha \mathbf{u} + \beta \rho^n e^{in\theta} \mathbf{v} + \gamma \rho^n e^{-in\theta} \mathbf{w})[1] \in J$

iff  $1/3 \alpha + (-1 - \sqrt{3}i) \rho^n e^{in\theta} \beta + (-1 + \sqrt{3}i) \rho^n e^{-in\theta} \gamma > 1/3$

iff  $\sqrt{3} \sin(n\theta) > \cos(n\theta)$

# Model Checking of MC vs distribution logic

$$M = \begin{pmatrix} 0.6 & 0.1 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}$$

Init:  $(1/4, 1/4, 1/2)$

Discretization  $I=[0, 1/3], J=(1/3, 1]$ .

We have:  $M^n \text{init}[1] \in J$  iff  $\sqrt{3} \sin(n\theta) > \cos(n\theta)$

**By contradiction: if trajectory is ultimately periodic**, let  $k$  be a period after  $r$  first steps.

Now use  $\{kn\theta \bmod 2\pi \mid n \in \mathbb{N}\}$  is **dense** in  $[0, 2\pi]$  cause

$\theta = \cos^{-1}(4/\sqrt{19})$  is not a rational multiple of  $\pi$  (using algebraic integers).

$\Rightarrow$  Can find  $n, n' > r$  such that  $\sqrt{3} \sin(kn\theta) > \cos(kn\theta)$  and  $\sqrt{3} \sin(kn'\theta) < \cos(kn'\theta)$ .

Hence  $M^{kn} \text{init}[1] \in J$  but  $M^{kn'} \text{init}[1]$  not in  $J$ , contradiction with ultimate periodicity

**$\rightarrow$  Hence the language  $L_{M^{\text{init}}}$  is not regular.**

# Results on distribution based logics:

$L_{M^{\text{init}}}$  is not regular in general even with 3 states.

[Agrawal, Akshay, G., Thiagarajan, JACM'14]

$L_{M^{\text{init}}}$  is regular for 2 states.

**Conjecture:**  $L_{M^{\text{init}}}$  regular if all eigen values are roots of real number (and distincts). Not easy for set of initial distrib.

[Ouaknine-Worrel'14]:

«Eventually always  $X[i] \geq p$ » is decidable for Markov Chains with 6 states

(comes from decidability of ultimate positivity of the Skolem problem).

In case all eigen values of  $M$  are distinct,  
decidable for all Markov Chains.

# Sum-up of trajectories for irred. aper. chains:

Discretization:  $[0, 1/2), [1/2, 1]$

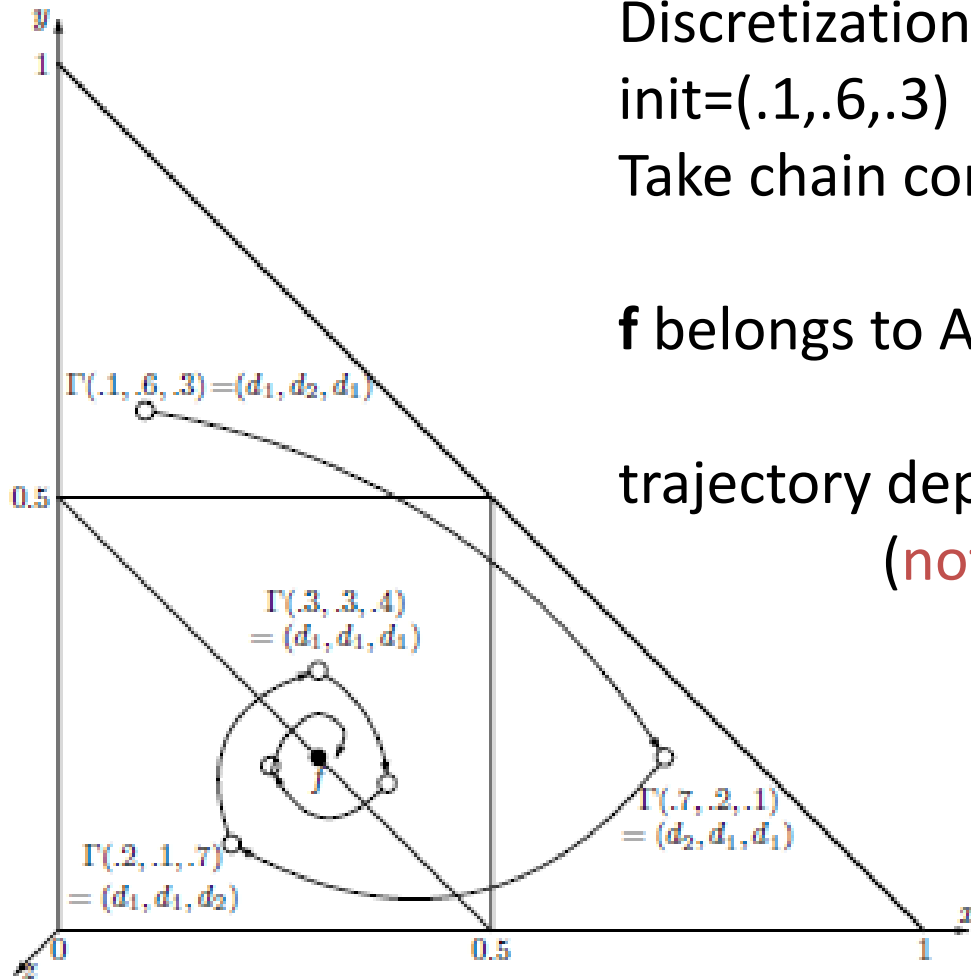
init =  $(.1, .6, .3)$

Take chain converging towards  $\mathbf{f} = (1/3, 1/6, 1/2)$

$\mathbf{f}$  belongs to  $A = ([0, 1/2), [0, 1/2), [1/2, 1])$

trajectory depicted: CDABBA...

(not ultimately periodic in general)



$\mathbf{f}$  at distance 0 of  $B = ([0, 1/2), [0, 1/2), [0, 1/2))$

Approximation for Markov Chains.



# Approximations for irreducible aperiodic chains:

Fix  $\epsilon \Rightarrow K$  such that  $|M^k \mathbf{u} - \mathbf{f}| < \epsilon$  for all distribution  $\mathbf{u}$ .

$A_1 \dots A_n$  is an  $\epsilon$  approximate symbolic trajectory of a concrete distribution trajectory  $d_1 \dots d_n$  if  $d_i \in A_i$  for all  $i < K$  and  $d_i$  is  $\epsilon$  close to  $A_i$  for  $i > K$ .

Exact symbolic trajectory from init: CDABBA...

$\epsilon \Rightarrow K=4$ ,

Approx symbolic trajectories:

CDABAA..., CDABAB..., CDABBA..., CDABBB....

$\Rightarrow$  CDAB (A or B)\* is regular.

# Approximations for irreducible aperiodic chains:

Th: Given MC + Init (set), it is decidable [AAGT, LICS'12] whether:

If for **some concrete** trajectory  $w$ , there **does not exist** a **symbolic approx** trajectory satisfying  $\phi$ , then  $w$  does not satisfy  $\phi$ .

$\Rightarrow$  system does not satisfy  $\phi$ .

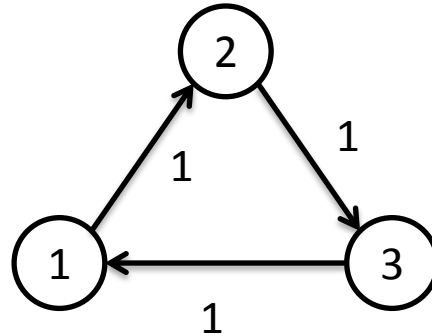
If for **all concrete** trajectory  $w$ , **all symbolic approx** trajectories satisfying  $\phi$ , then all  $w$  satisfies  $\phi$ .

$\Rightarrow$  system satisfies  $\phi$ .

Undetermined: for all concrete trajectory, there exists symbolic approx satisfying  $\phi$ , but not for all.

$\Rightarrow$  Refine  $\epsilon$  to reduce number of approx trajectories.

# Irreducible Periodic chains



M is **periodic** of period 3.

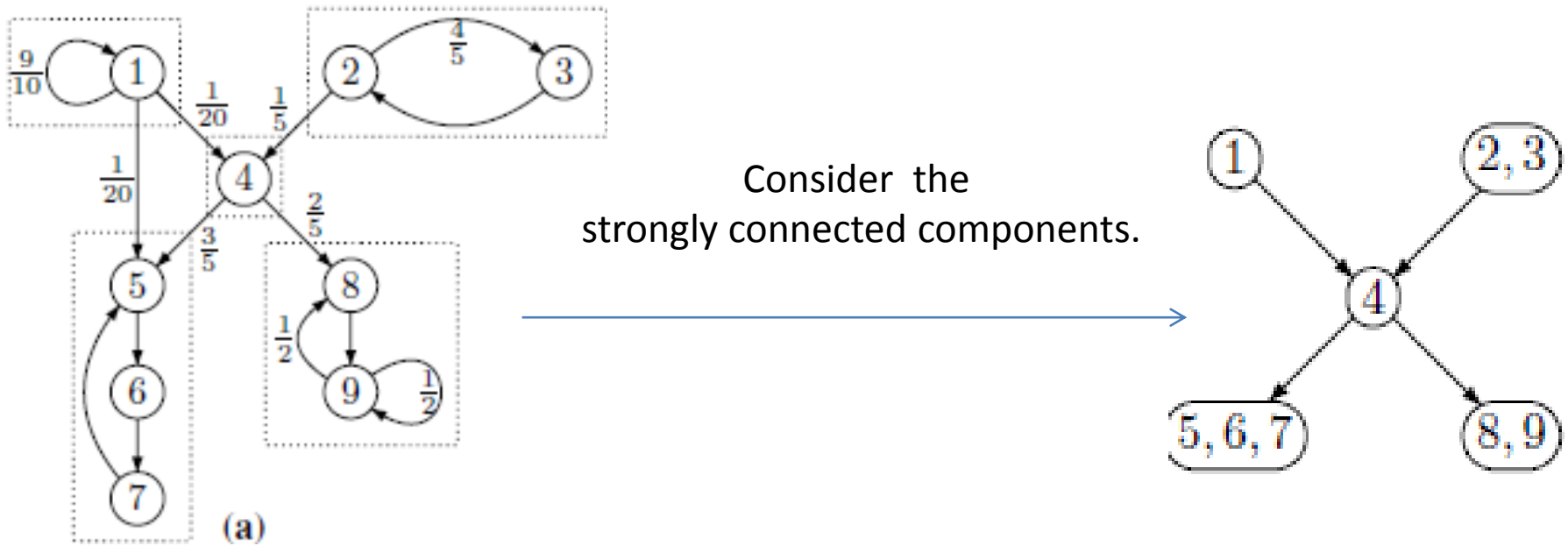
$M^3$  is irreducible aperiodic on partition of nodes.

Consider  $M^3$  from **Init**,

Consider  $M^3$  from **M Init**,

Consider  $M^3$  from  **$M^2$  Init**

# Not irreducible chains



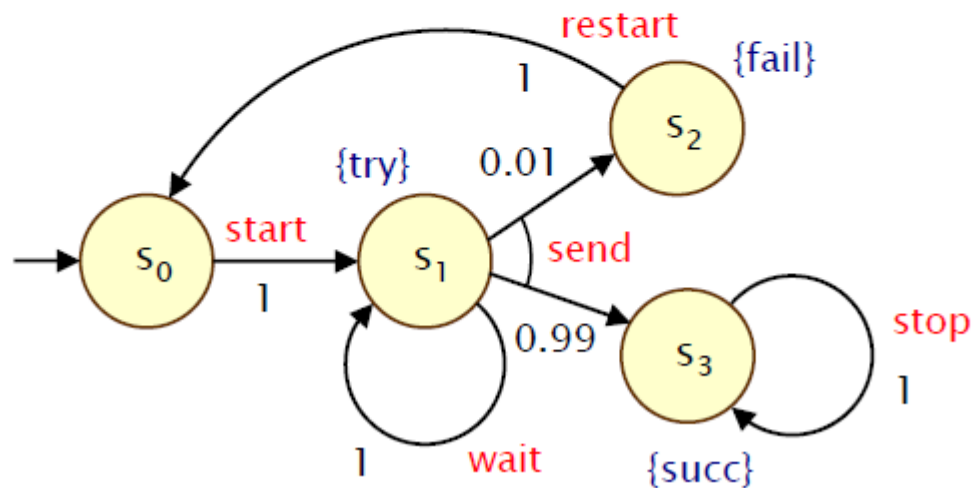
Stationary distributions have weight 0 for non bottom SCC (1; 2-3, 4).  
⇒ Analyse the bottom SCC with earlier algorithm.

**Tough part:** Analyse non bottom SCC to get weights for bottom SCC, depending on Initial distribution (algorithm close to CTL model checking)

Markov Decision Process?

# Markov Decision Process (MDP)

- after one step, process **starts** trying to send a message
- then, a nondeterministic choice between: (a) **waiting** a step because the channel is unready; (b) **sending** the message
- if the latter, with probability 0.99 send **successfully** and **stop**
- and with probability 0.01, message sending **fails**, **restart**



# Markov Decision Process (MDP)

Can be seen as 1 and a  $\frac{1}{2}$  player game:

$\frac{1}{2}$  player is the random player, which plays according to the probabilities.

The other player is either demonic (want to break the property), or angelic (want to satisfy it)

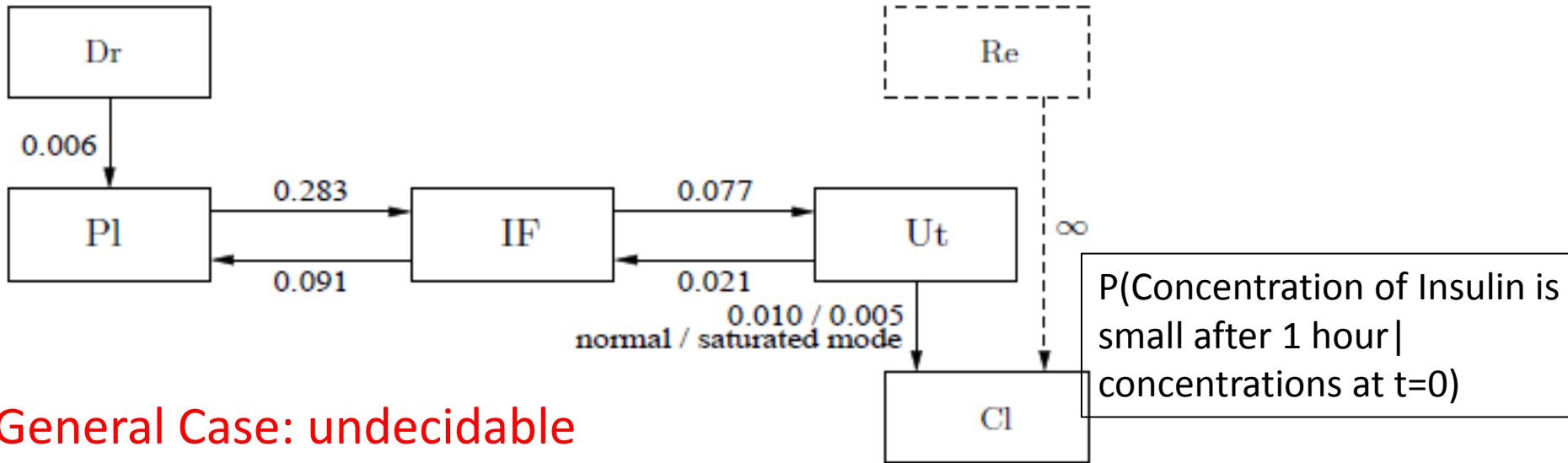
A particular set of choices of the players are called scheduler or strategies.

MDP + scheduler = Markov Chain.

- Decide for all scheduler, does a property hold? or
- Does there exists a scheduler such that a property holds?

Or equivalently, determine  $\text{Prob}^{\max}$  and  $\text{Prob}^{\min}$ ....

# MDP + distribution based:



General Case: undecidable

[Chadha et al Qest'11]

Choices are made based on time, not based on the run

=>

Corresponds to blind POMDP (= Rabin's PFA).

undecidable

[Paz, Berttoni... 70s]

$$N = \begin{bmatrix} 0.94 & 0.02434 & 0.02567 & 0.00798 & 0.00024 & 0 \\ 0.00 & 0.20724 & 0.48298 & 0.29624 & 0.01353 & 0 \\ 0.00 & 0.15531 & 0.42549 & 0.39520 & 0.02400 & 0 \\ 0.00 & 0.02598 & 0.10778 & 0.77854 & 0.08770 & 0 \\ 0.00 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0 \\ 0.00 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 0.94 & 0.02435 & 0.02568 & 0.00809 & 0.00012 & 0 \\ 0.00 & 0.20728 & 0.48329 & 0.30257 & 0.00686 & 0 \\ 0.00 & 0.15540 & 0.42612 & 0.40626 & 0.01221 & 0 \\ 0.00 & 0.02653 & 0.11080 & 0.81776 & 0.04491 & 0 \\ 0.00 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0 \\ 0.00 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0 \end{bmatrix}$$



# Concluding Remarks

# Perspectives

Things to do:

extend from MC to MDPs...

What about interval MCs?

compute exact bound on errors (bound on contracting factor)...

Other approximations schemes:

« Decidable in good complexity for almost all instances. » ?

Upper and lower approximation of the Markov Chain by a class with good algorithm (« class dense in the space of MC »).

=> Deal with an inbetween language.

Thank You!