

Reinforcement Planning for effective ε -optimal policies in dense time with discontinuities

Léo Henry ✉ 

University College London, UK

Blaise Genest ✉ 

CNRS and CNRS@CREATE, IPAL, France and Singapore

Alexandre Drewery ✉

ENS Rennes, France

Abstract

Lately, the model of (Decision) Stochastic Timed Automata (DSTA) has been proposed, to model those Cyber Physical Systems displaying dense time (physical part), discrete actions and discontinuities such as timeouts (cyber part). The state of the art results on controlling DSTAs are however not ideal: in the case of infinite horizon, optimal controllers do not exist, while for timed bounded behaviors, we do not know how to build such controllers, even ε -optimal ones.

In this paper, we develop a theory of Reinforcement Planning in the setting of DSTAs, for discounted infinite horizon objectives. We show that optimal controllers do exist in general. Further, for DSTAs with 1 clock (which already generalize Continuous Time MDPs with e.g. timeouts), we provide an effective procedure to compute ε -optimal controllers. It is worth noting that we do *not* rely on the discretization of the time space, but consider symbolic representations instead. Evaluation on a DSTA shows that this method can be more efficient. Last, we show on a counterexample that this is the furthest this construction can go, as it cannot be extended to 2 or more clocks.

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1 Introduction

Controlling *Cyber Physical Systems* (CPSs for short) is an important task to optimize resources of today's complex systems. However, it is also challenging, as CPSs often mix discrete control, continuous state space, time and stochasticity, between others. A well-studied model of CPSs is the one of continuous-time Markov Decision Processes (CTMDPs for short). On the theoretical side, optimal controllers exist in the timed-bounded horizon setting [BHKH05, RS13, BFK+09], with infinite horizon [Ber05, GH09], or with discounted infinite horizon [BD94]. However, building such optimal controller might not be as simple, even in the time bounded case [MSS20]. On the practical side, there are many algorithms to produce ε -optimal controllers, e.g. [RS13, BFK+09]. CTMDPs are fully continuous models that have been expanded in the previous decade by *Decision Stochastic Timed Automata* (DSTA, see [BS12, BBG14]), able to express hard real-time constraints (e.g. timeouts, deadlines) using real-valued clocks and guards on transitions stemming from the *Timed Automata* [AD94] and *Stochastic TAs* [BBM07] formalisms. In terms of DSTAs, CTMDPs are DSTAs with a single clock reset on all transitions and no guards.



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Theoretically, this ability to model hard constraints is very challenging as it produces discontinuities. It breaks the proof for the existence of optimal controller for CTMDPs and to produce ε -optimal policies in practice (except in the special case of value 1 [BBG14]). In [BS12], an existential theorem for optimal policies for timed bounded horizons in DSTA has been proved in a *non*-constructive manner. However, besides the fact that they can be chosen positional, little is known on the kind of policies necessary to achieve optimality.

In this paper, we develop a *Reinforcement Planning (RP)* framework [SB98] to prove (Theorem 8) the existence of optimal positional policies for DSTA in general for unbounded horizon equipped with a discount. This is the best possible result, as optimal policies do not exist in general for unbounded horizon without discount [BS12].

Our main results concern ε -approximations: we prove that for one-clock DSTAs, which are already more powerful than CTMDPs (the unique clock needs not be reset at every transition) and allows to model discontinuous hard constraints such as timeouts, *polytope policies* (defined hereafter) suffices. A polytope policy partitions $[0, \infty)$ (value of the clock) with a finite number of intervals and plays one action on each interval. Technically, we develop a symbolic value iteration algorithm, creating polytopes only when necessary using *symbolic functional analysis* (Lemma 19); and then use *numerical analysis* to effectively compute approximations of these polytopes (Lemma 21). Last, we show that this is the furthest this method can go, as in the presence of 2 clocks or more, the optimal policy after n iterations is not a polytope policy (Theorem 10). Omitted proofs can be found in appendix.

Related Work: Reinforcement *Planning* [SB98, Doya00] provides foundations to reason about observations for completely specified models, that ground Reinforcement *Learning* (e.g. Monte-Carlo method) which tackle un- or semi-specified systems. RP provides implementable algorithms for fully specified models, which is also the case of our methodology, although the implementation would require a heavy use of both symbolic functional as well as numerical analysis tools which are very technical and tedious to implement. RP has first been theorized for discrete models with finite state space (MDPs) [SB98]. Some version exist for a totally continuous world (continuous state space, continuous action, continuous behavior) [Doya00]. Some RP frameworks have been developed for some specific model with discrete actions and continuous state space: CTMDPs [BD94], ODEs [YHL21], but without any discontinuous behaviors such as deadline or timeouts. We did not find any RP methodology which can be applied to a setting mixing continuous state space, discrete action set and non-continuous behaviors (timeouts, deadlines). The closest related work is [CHK+11], where the authors design an effective algorithm to produce ε -optimal policies. The main differences are that the method is very different, discretizing time in many modes of ϵ -size (depending on the ε -precision desired), whereas we compute a minimal number of (larger) modes using symbolic function analysis; and the model considered is different, namely Markov Timed Automata (MTA). Compared with DSTA, MTA cannot model urgency: time can always elapse, and all distributions are exponential, whereas DSTA can enforce leaving a state before some time-out using uniform distributions. Further, uniform distributions are harder to handle than exponential ones. It is thus unclear how to lift similar time-discretization bounds for DSTA. We numerically evaluate on a DSTA the efficiency of our technique in Section 5.3 and compare with the complexity of time-discretization for a MTA of similar size.

2 Decision Stochastic Timed Automata

We start by defining the model of CPS we are considering in the paper, namely Decision Stochastic Timed Automata (*DSTA* for short), extending CTMDPs with hard constraints.

2.1 Notations

We fix finite sets \mathcal{C} of clocks, and \mathcal{V} of *valuations over \mathcal{C}* , i.e. the set of all maps of $\mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$ (clocks always have non-negative values). Given a valuation $v \in \mathcal{V}$, we define two operations:

Time elapse $v + \delta$ with $\delta \in \mathbb{R}_{\geq 0}$ is a valuation such that for all $x \in \mathcal{C}$ $(v + \delta)(x) = v(x) + \delta$;

Reset $\mathcal{C}' \subseteq \mathcal{C}$ $v_{[\mathcal{C}' \leftarrow 0]}$ is the valuation $x \mapsto 0$ if $x \in \mathcal{C}'$ and $x \mapsto v(x)$ otherwise.

We will sometimes consider a valuation v as a vector in $\mathbb{R}_{\geq 0}^{|\mathcal{C}|}$.

We denote \mathcal{G} the set of guards, i.e. of constraints of the form $\bigwedge_{1 \leq i \leq |\mathcal{C}|} (x_i \in I_i)$ and \mathcal{I} the set of *invariants*, i.e. guards featuring intervals with no lower bound. The set of *constraints over a set of clocks \mathcal{C}* is the set of all constraints of the form $(\sum_{1 \leq i \leq |\mathcal{C}|} a_i \times x_i) \bowtie q$ with $a_i \in \mathbb{Q}$, $x_i \in \mathcal{C}$, I_i an interval of \mathbb{Q} , $\bowtie \in \{<, \leq, =, \geq, >\}$ and $q \in \mathbb{Q}$. A *polytope* $z \in \mathcal{Z}$ is a finite set of constraints over \mathcal{C} . Let $z \in \mathcal{Z}$ and $v \in \mathcal{V}$. We write $v \models z$ if and only if v satisfies all constraints in z . A polytope z defines a subset $\mathcal{V}(z)$ of $\mathbb{R}^{|\mathcal{C}|}$ containing the valuations v such that $v \models z$. We define $z_{[\mathcal{C}' \leftarrow 0]}$ as the polytope such that $\mathcal{V}(z_{[\mathcal{C}' \leftarrow 0]}) = \{v_{[\mathcal{C}' \leftarrow 0]} \mid v \in \mathcal{V}(z)\}$.

For a function P from some set to booleans we note $1_P : y \mapsto 1$ iff $P(y)$ and otherwise 0. We write $\mathcal{D}(Y)$ for the set of all probabilities distributions over a set Y .

2.2 Decision Stochastic Timed Automata

We consider DSTAs [BS12, BBG14], transition systems defining probabilities on sets of runs by assigning probability distributions to delays, up to a policy dictating the transitions taken.

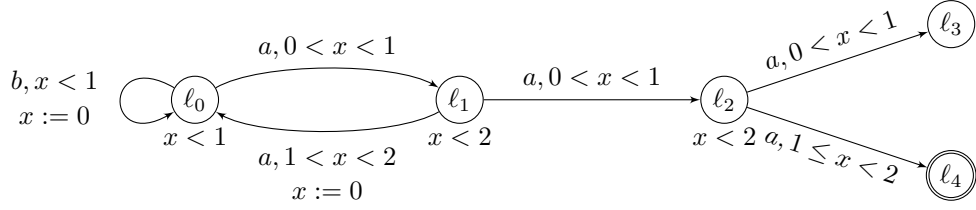
- **Definition 1.** A DSTA over clocks \mathcal{C} and actions A is a tuple $\mathcal{A} = (\mathcal{L}, E, \ell_0, I, \mu)$ with
- \mathcal{L} a set of control state and $\ell_0 \in \mathcal{L}$ is the initial location,
 - E a set of transitions of the form $(\ell_1, a, g, r, \ell_2) \in \mathcal{L} \times A \times \mathcal{G} \times 2^{\mathcal{C}} \times \mathcal{L}$, where g is the guard of the transition and r is the finite set of clocks to reset when taking the transition,
 - $I : \mathcal{L} \rightarrow \mathcal{I}$ defines the invariants the clock values need to satisfy to stay in a control state,
 - $\mu = (\mu_{(\ell, v)})_{(\ell, v) \in \mathcal{L} \times \mathcal{V}(X)}$ defines probability distributions over delays for each control state and clock valuation. For $I(\ell, v) = \{\delta \in \mathbb{R}^+ \mid v + \delta \models I(\ell)\}$, it is of one of the following type:
 - if $I(\ell, v)$ is the singleton $\{0\}$, then μ is the Dirac distribution $\mu(0) = 1$,
 - otherwise, if $I(\ell, v)$ is bounded, then μ is the uniform distribution over $I(\ell, v)$,
 - otherwise, $I(\ell, v) = [0, \infty)$ and μ is an exponential distribution $\mu(\delta) = e^{-\alpha_\ell \delta}$ for some parameter $\alpha_\ell \in \mathbb{R}_{>0}$ independent over the valuation v .

Intuitively, a move in a DSTA \mathcal{A} from a configuration $(\ell, v) \in \mathcal{L} \times \mathcal{V}$ is done in two steps: first, a delay δ is randomly chosen according to the distribution $\mu_{(\ell, v)}$, and then the policy chooses a transition $e = (\ell, a, g, r, \ell')$ enabled from $(\ell, v + \delta)$, reaching $(\ell', (v + \delta)_{[r \leftarrow 0]})$.

Formally, we define a *partial run* of a DSTA as a sequence of transitions and delays $\rho = (e_i, \delta_i)_{1 \leq i \leq n}$ starting from a configuration (ℓ_1, v_1) such that a corresponding sequence of configurations can be defined: $\rho = ((\ell_i, v_i) \xrightarrow{e_i} (\ell_{i+1}, v_{i[r_i \leftarrow 0]}) \xrightarrow{\delta_i} (\ell_{i+1}, v_{i[r_i \leftarrow 0]} + \delta_i))_{1 \leq i \leq n}$ (notably, the transitions e_i are enabled in (ℓ_i, v_i)). A *run* is a partial run starting in the initial configuration $(\ell_0, \mathbf{0})$. We note $\text{Runs}(\mathcal{A})$ (resp. $\text{PRuns}(\mathcal{A}, (\ell, v))$) the set of runs (resp. partial runs) of a DSTA \mathcal{A} (resp. starting from a configuration (ℓ, v)).

To start with a random passing of time, we can add a dummy state **init** with no incoming transition. Every action from **init** leads to ℓ_0 , resetting all clocks, so that any random delay can be taken in **init**. We will not depict such a state **init** in the different figures.

- **Example 2.** Figure 1 depicts DSTA \mathcal{A} . In \mathcal{A} , the user can play b in ℓ_0 to roll for a new clock value. When they play a , they are sent to ℓ_1 . Depending on the delay randomly chosen in ℓ_1 , the control states either goes back to ℓ_0 ($x > 1$) and resets the clock x , or it moves to



■ **Figure 1** A DSTA with one clock x with initial state ℓ_0 . The goal is to reach ℓ_4 .

ℓ_2 ($x < 1$) from which another delay is randomly drawn and added to x : if this delay makes $x > 1$, then the run goes to winning state, else it goes to a losing state. Intuitively, playing a from ℓ_0 with x very close to 1 is optimal, as then the chance to get stuck in the losing state is very small (the sum of the delays from ℓ_1 and ℓ_2 needs to be smaller than $1 - x$).

To discuss the semantics of DSTAs, we have to define the probabilities some behaviors have to appear. As (almost) every run has a null probability to appear, we discuss at the level of (partial) *paths*, i.e. sequences of transitions. A path $\pi = (e_i)_{1 \leq i \leq n}$ represents the set of runs ρ such that there exist delays $(\delta_i)_{1 \leq i \leq n}$ with a run $\rho = (e_i, \delta_i)_{1 \leq i \leq n}$. We say that a run ρ *models* a path $\pi \in E^*$ when their sequences of transitions match. A partial path is said *feasible* when it has a model. We call *path* a partial path that is modeled by a (initial) run. For paths, we will only consider (initial) runs as models. We write $PPaths(\mathcal{A})$ (resp. $Paths(\mathcal{A})$) for the set of partial paths (resp. paths) of an automaton \mathcal{A} .

2.3 Rewards and policies

We consider a discounted average reward model as the objective function (as usual in Reinforcement Planning), by contrast with winning state and infinite horizon (reward = 1 if winning state ever reached, 0 otherwise) or finite horizon (reward = 1 if winning state reached before the time horizon, 0 otherwise). More precisely, each transition is associated with a reward, and we consider the *discounted sum* of all rewards as our objective function, with immediate rewards of greater interest than later ones. Discount allows to tackle unbounded runs, while optimal policies do not exist for unbounded horizons without discount [BS12].

► **Definition 3.** Consider a DSTA $\mathcal{A} = (\mathcal{L}, E, \ell_0, I, \mu)$. A reward function is a function $\mathcal{R} : E \rightarrow \mathbb{N}$ associating a reward to each transition. Given discount $0 < \gamma < 1$, the discounted reward of a run $\rho = (e_i, v_i)_i$ or of a path $\pi = (e_i)_i$ is $\mathcal{R}(\rho) = \mathcal{R}(\pi) = \sum_i \gamma^{i-1} \mathcal{R}(e_i)$.

The rewards associated with Figure 1 are 0 on every transition except on from ℓ_2 to ℓ_4 , where the reward is 1. Hence a path reaching ℓ_4 in i steps will get discounted reward γ^i .

The decisions made by the agent are formalized as *policies*. Policies depend on the type of input they can use to take a decision: *generic policies* can use any information about the history of the run to make stochastic choices, while more restrictive *pure positional policies* can only use the current configuration and decide on a given transition deterministically.

► **Definition 4.** A policy σ on a DSTA $\mathcal{A} = (\mathcal{L}, E, \ell_0, I, \mu)$ is a function $\sigma : Runs(\mathcal{A}) \rightarrow \mathcal{D}(E)$. A pure positional policy is a policy σ such that for all pairs of runs ρ, ρ' ending in the same configuration $(\ell, v) \in \mathcal{L} \times \mathcal{V}$, we have $\sigma(\rho) = \sigma(\rho')$ is a Dirac distribution in one action. We can thus represent any pure positional policy as a function $\sigma : (\mathcal{L} \times \mathcal{V}) \rightarrow E$.

Given a policy σ and a history $\rho \in \text{Runs}(\mathcal{A})$ ending in (ℓ, v) , we can define the probability $P_\sigma(\pi \mid \rho)$ for the behavior of the DSTA to model the path $\pi = (e_i)_{1 \leq i \leq n}$ after ρ as follows:

$$P_\sigma(\pi \mid \rho) = \sigma(\rho)(e_1) \int_{\delta \in I(\ell_1, v_{[r_1 \leftarrow 0]})} P_\sigma((e_i)_{2 \leq i \leq n} \mid \rho \cdot (e_1, \delta)) d\mu_{(\ell_1, v_{[r_1 \leftarrow 0]})}$$

with ℓ_1 the target location of e_1 and r_1 the set of clocks it resets. Given a reward function \mathcal{R} this can be used to assign a *value* to a policy.

► **Definition 5.** Fix a reward function \mathcal{R} on a DSTA \mathcal{A} with discount γ and an integer $n \in \mathbb{N}$. The value of a policy σ after a history $\rho \in \text{Runs}(\mathcal{A})$ for n steps is

$$\text{Val}_\sigma^n(\rho) = \sum_{\pi=(e_i)_{1 \leq i \leq n} \in E^n} P_\sigma(\pi \mid \rho) \mathcal{R}(\pi) .$$

The value of a policy for history ρ is $\text{Val}_\sigma(\rho) = \lim_{n \rightarrow \infty} \text{Val}_\sigma^n(\rho)$. We write $\text{Val}^*(\rho) = \sup_\sigma \text{Val}_\sigma(\rho)$ for the optimal value. The value $\text{Val}_{(\sigma)}^*(\mathcal{A}) = \text{Val}_{(\sigma)}^*(\lambda)$, denoted $\text{Val}_{(\sigma)}^*$, is the value associated with the empty run. A policy σ^* is said optimal when $\text{Val}^* = \text{Val}_{\sigma^*}$.

We now prove that rewards are bounded with known bounds.

► **Lemma 6.** Let $\text{MaxReward} = \max_{e \in E} \mathcal{R}(e)$ the maximal reward associated with a transition. Then for any path π , $0 \leq \mathcal{R}(\pi) \leq \frac{\text{MaxReward}}{1-\gamma}$ and similarly for runs.

Proof. $0 \leq \mathcal{R}(\rho)$ simply because $\mathcal{R} : E \rightarrow \mathbb{N}$. Furthermore, $\mathcal{R}((e_i)_{1 \leq i \leq n}) = \sum_{i=1}^n \gamma^{i-1} \mathcal{R}(e_i) \leq \sum_{i=1}^{\infty} \gamma^{i-1} \text{MaxReward}$ which entails the upper bound. ◀

Similarly, the limit of $\text{Val}_\sigma^n(\rho)$ is well defined, as:

$$\begin{aligned} 0 \leq \text{Val}_\sigma^{n+1}(\rho) - \text{Val}_\sigma^n(\rho) &= \sum_{(e_i)_{1 \leq i \leq n+1}} P_\sigma((e_i) \mid \rho) \gamma^n \mathcal{R}(e_{n+1}) \\ &\leq \mathcal{R}^* \sum_{(e_i)_{1 \leq i \leq n+1}} P_\sigma((e_i) \mid \rho) \gamma^n \\ &= \gamma^n \text{MaxReward} \end{aligned}$$

which converges to 0 when n tends toward the infinite. Thus values are well defined. One can also see that the values are bounded in the same way as rewards. Interestingly, one can see that values for n steps *converge uniformly* towards values, which will be useful for latter proofs. We write MaxReward (resp. MaxValue) for the supremal reward (resp. value).

Similarly, we adopt the notation $\text{Val}_\sigma(\rho, e)$ (resp. $\text{Val}^*(\rho, e)$) to express the value of first playing e and then following σ (resp. having an optimal value). Finally, for (pure) positional policy, we replace ρ by a configuration (ℓ, v) (e.g. $\text{Val}_\sigma(\ell, v)$). We say that a policy σ is ε -optimal whenever $\text{Val}_\sigma(\mathcal{A}) \geq \text{Val}^*(\mathcal{A}) - \varepsilon$.

In general, one cannot represent finitely a measurable policy, nor even a pure positional policy even for a DSTA with a unique clock. We now define a class of policies that can be finitely described, called *polytopes policies* [BS12]. Given a control state ℓ , it partitions \mathcal{V} with a finite number of polytopes and assigns a unique action to each polytope.

► **Definition 7.** Let \mathcal{A} be a DSTA. A pure positional policy σ is a polytope policy if for each location ℓ of \mathcal{A} there exists a finite set of polytopes Z_1, \dots, Z_n that partition \mathcal{V} such that $\forall 1 \leq i \leq n, \forall v, v' \in Z_i, \sigma(\ell, v) = \sigma(\ell, v')$.

3 Main Results

Our first contribution is to show, by adapting the RP framework to DSTA (see Section 4, which serves as proof of Theorem 8), that *pure positional* optimal policies exist:

► **Theorem 8.** *Let \mathcal{A} be a DSTA. Then there exists a pure positional policy σ^* such that for every policy σ and every run ρ , $Val_{\sigma^*}(\rho) \geq Val_{\sigma}(\rho)$.*

Notice that discounted infinite horizon is the most generic reward scheme possible enjoying the existence of optimal policies, as it has been shown in [BS12] that optimal policies do not exist in general for undiscounted infinite horizon. This matches the existence of optimal policies for finite horizon properties of [BS12], with a vastly different proof technique.

Concerning ε -optimal policies, we show that the class of *polytope policies*, which are finitely describable pure positional policies, are sufficient for DSTA with a unique clock $\mathcal{C} = \{x\}$. The main difficulty is that there are *uncountably many* configurations, and from each a (pure positional) policy needs to provide which action to play. We do so by developing a *symbolic* value iteration algorithm and prove in Lemma 19 that the policies generated are polytope policies, using symbolic functional analysis. Even harder is to actually compute effectively a ε -approximation from the *uncountably many* configurations, and not only dealing with existence: Using numerical functional analysis, we explain how to compute an approximation of the finitely many positions of the changes of sign of the value function, without approximating the value function itself. This is our second and main contribution:

► **Theorem 9.** *Let \mathcal{A} be a DSTA with a unique clock and $\varepsilon > 0$. Then there exists an effectively constructible positional polytope policy σ_{ε} that is ε -optimal.*

Section 5 serves as proof of Theorem 9: Lemma 19 shows that there exist an *optimal* polytope policy σ_n for n steps of value iteration. This is complex as we do not know how to provide a close-form solution for the value function itself, and we conjecture that it is a hard problem. Eventually, after n iterations, σ_n will be $\frac{\varepsilon}{2}$ -optimal. Using numerical analysis, we show in Lemma 21 in Section 5 how to compute effectively a polytope policy σ whose value is at most $\frac{\varepsilon}{2}$ away from the value of σ_n , providing us with a polytope policy that is ε -optimal. This is in stark contrast with the non-constructive proof of [BS12], which needs uncountable time to produce ε -optimal policies.

According to Lemma 19, polytope policies are optimal for DSTA with 1 clock and without loops, as the value is met after a finite number of steps of value iteration. Our third contribution is that polytope policies are not optimal with 2 clocks and without loops.

► **Theorem 10.** *There is a DSTA \mathcal{A} with 2 clocks and without loops such that for all polytope policies σ , there exists a policy σ' with $Val_{R,\gamma}(\mathcal{A}, \sigma) < Val_{R,\gamma}(\mathcal{A}, \sigma')$.*

The proof can be found in Section 5.7. It is inspired by [BS12], where it is shown polytope policies are not optimal for finite horizon properties, even for DSTA with a *unique* clock. We use the second clock to simulate the bound on the time horizon. Theorem 9 shows that for one clock, the setting of discounted infinite horizon has much more desirable properties than the one of bounded time. Notice that polytope policies could still be ε -optimal in general.

4 Bellman equations in dense time

We extend Reinforcement Planning [SB98] to DSTA models. We first focus on the introduction of Bellman Equations for this continuous setting. The Bellman Equations are the theoretical basis of RP, as they allow to incrementally compute or approximate the (optimal) values.

4.1 General policies

We first reason on general policies in order to prove the optimality of the *pure positional policies*. Once their optimality is proved, the discussion is restricted to this simpler class. We first define the Bellman Equations in their most general form for DSTA.

► **Proposition 11.** *Given a policy σ and an history ρ ending in configuration (ℓ, v) , $Val_\sigma(\rho)$ is the only bounded solution of the **Bellman Equation in dense time**:*

$$Val_\sigma(\rho) = \sum_{e=(\ell, a, g, r, \ell') \in E} \sigma(\rho)(e) \times \left(\mathcal{R}(e) + \gamma \times \int_{\delta \in \mathcal{I}(\ell', v_{[r \leftarrow 0]})} Val_\sigma(\rho \cdot (e, \delta)) d\mu_{(\ell', v_{[r \leftarrow 0]})} \right)$$

In a similar fashion, $Val_\sigma(\rho, e)$ for $e = (\ell, a, g, r, \ell')$ is the unique solution of:

$$Val_\sigma(\rho, e) = \mathcal{R}(e) + \gamma \times \int_{\delta \in \mathcal{I}(\ell', v_{[r \leftarrow 0]})} \sum_{e'=(\ell', a, g, r, \ell'') \in E} \sigma(\rho)(e') \times Val_\sigma(\rho \cdot (e, \delta), e') d\mu_{(\ell', v_{[r \leftarrow 0]})}$$

Proof sketch. We prove that the value is a solution by using the definition of Val as the limit of Val^n and unfolding it (taking the first action out of the limit). The uniqueness of the bounded solution is proved by comparing two potentially different one: $F(\rho) - F'(\rho)$. Using the equation allows to reduce their difference step by step as the first reward does not depend on F / F' . After n steps, the difference is bounded by γ^n times a probability distribution of differences. As the functions are bounded, taking the limit proves the uniqueness. ◀

From Bellman's equations, the following relations can be inferred:

► **Corollary 12.** *Given a policy σ and a run ρ ending in (ℓ, v) :*

$$Val_\sigma(\rho) = \sum_{e \in E} \sigma(\rho)(e) \cdot Val_\sigma(\rho, e) \quad \text{and} \quad Val_\sigma(\rho, e) = \mathcal{R}(e) + \gamma \int_{\delta \in \mathcal{I}(\ell', v_{[r \leftarrow 0]})} Val_\sigma(\rho(e, \delta)) d\mu_{(\ell', v_{[r \leftarrow 0]})}$$

with r the reset of e and ℓ' its target.

For any policy, we can construct a pure positional policy at least as efficient:

► **Proposition 13.** *For any policy σ , there exists a pure positional policy σ' such that $Val_{\sigma'} \geq Val_\sigma$. In particular one such policy can be defined from σ in the following way: consider a history ρ ending in configuration (ℓ, v) , then*

$$\sigma'(\ell, v) = \arg \max_{e \in E} \sup_{\rho' \text{ ending in } (\ell, v)} \left(\mathcal{R}(e) + \gamma \int_{\delta \in \mathcal{I}(\ell', v_{[r \leftarrow 0]})} Val_\sigma(\rho' \cdot (e, \delta)) d\mu_{(\ell, v_{[r \leftarrow 0]})} \right)$$

Proof sketch. Using Bellman Equation on Val_σ allows to see that it is lower than what is obtained by playing optimally at the next step (on all executions) and then following σ . Applying the same reasoning n times allows to show that it is lower than Val_σ^n , plus a bounded term depending of Val_σ discounted by γ^n . Taking the limit then allows to conclude. ◀

4.2 Positional policies

As we now know from Proposition 13 that positional policies are as efficient as general policies, the rest of this paper mentions only pure positional policies. We will use the Bellman Equation in their restricted form for pure positional policies, provided now:

$$\text{Val}_\sigma(\ell, v) = \mathcal{R}(\sigma(\ell, v)) + \gamma \int_{\delta \in \mathbf{I}(\ell', v_{[r \leftarrow 0]})} \text{Val}_\sigma(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})}$$

with ℓ' the target location of $\sigma(\ell, v)$.

$$\text{Val}_\sigma(\ell, v, e) = \mathcal{R}(e) + \gamma \int_{\delta \in \mathbf{I}(\ell', v_{[r \leftarrow 0]})} \text{Val}_\sigma(\ell', v_{[r \leftarrow 0]} + \delta, \sigma(\ell', v_{[r \leftarrow 0]} + \delta)) d\mu_{(\ell', v_{[r \leftarrow 0]})}$$

with $e(l, x, a) = (l, a, g, r, l')$. The Bellman Equations rely on the integral of the value on the interval of possible delays. We call this the *interval value* function, and write $\text{IVal}_\sigma(\ell, v) = \int_{\delta \in \mathbf{I}(\ell', v_{[r \leftarrow 0]})} \text{Val}_\sigma(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})}$. We have $\text{Val}_\sigma = \mathcal{R}(\sigma(\ell, v)) + \gamma \cdot \text{IVal}_\sigma(\ell', v_{[r \leftarrow 0]})$.

We can show that if locally deviating from a value function according to a policy always gives a better result, then globally changing to the new policy enhances the expected return.

► **Proposition 14.** *Consider a function F such that for all configurations (ℓ, v) $F(\ell, v, e) = \mathcal{R}(e) + \gamma \int_{\delta \in \mathbf{I}(\ell', v_{[r \leftarrow 0]})} F(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})}$ with ℓ' the target of e . For a policy σ , if for all configurations (ℓ, v) , $F(\ell, v, \sigma(\ell, v)) \geq F(\ell, v)$ then $\text{Val}_\sigma \geq F$. Furthermore $\text{Val}_\sigma > F$ if and only if there exists a configuration where the inequality is strict.*

Proof sketch. The result is obtained as previously by unfolding the equation for n step before using the discount and the boundedness of functions to conclude. ◀

Corollary 17 apply this to obtain a greedy improvement from the current policy. We now state the *Bellman optimality equations in dense time*, characterizing the optimal values.

► **Proposition 15.** *Val^* is the only bounded solution of the **Bellman Optimality Equation**:*

$$\text{Val}^*(\ell, v) = \max_{e \in E} \left(\mathcal{R}(e) + \gamma \int_{\delta \in \mathbf{I}(\ell', v_{[r \leftarrow 0]})} \text{Val}^*(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})} \right)$$

with r the resets of e and ℓ' its target location. Similarly:

$$\text{Val}^*(\ell, v, e) = \mathcal{R}(e) + \gamma \int_{\delta \in \mathbf{I}(\ell', v_{[r \leftarrow 0]})} \max_{e \in E} \text{Val}^*(\ell, v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})}.$$

Proof sketch. The proof goes as for Bellman Equations: unfolding the definition of Val^* on one step allows to prove that it is a solution, while uniqueness is showed by comparing two solutions and using γ to limit their difference to 0 in the limit. ◀

We will again write $\text{IVal}^*(\ell, v) = \int_{\delta \in \mathbf{I}(\ell', v_{[r \leftarrow 0]})} \text{Val}^*(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})}$. As for values of policies, we use it to deduce relations between optimal values for configurations and transitions.

► **Corollary 16.** *For any configuration (ℓ, v) and transition $e \in E$ we have the following equalities: $\text{Val}^*(\ell, v) = \max_{e \in E} \text{Val}^*(\ell, v, e)$ and $\text{Val}^*(\ell, v, e) = \mathcal{R}(e) + \text{IVal}^*(\ell', v_{[r \leftarrow 0]})$*

Using these equations and Proposition 14, we obtain two important results: the policy improvement theory works as for MDPs (it has no local minima) and the pure positional policies are optimal.

► **Corollary 17.** *Given a policy σ , we can deduce a positional pure policy σ' such that $\forall(\ell, v) \in \mathcal{L} \times \mathcal{V}$, $\sigma'(\ell, v) = \arg \max_{e \in E} (\mathcal{R}(e) + \gamma \cdot IVal_{\sigma}(\ell', v_{[r \leftarrow 0]}))$ with r the reset of e and ℓ' its target. Then $Val_{\sigma'} \geq Val_{\sigma}$. Furthermore, if $Val_{\sigma'} = Val_{\sigma}$ then σ is optimal.*

We now restate Theorem 8 in this context and prove it:

► **Theorem 18.** *There exists a pure positional policy σ^* such that $Val_{\sigma^*} = Val^*$.*

Proof sketch. For each configuration, we consider the action which is associated with the maximal expected reward when being played from this configuration (chosen arbitrarily from the actions with the same expected maximal reward). We call such an action an optimal action. We consider the positional policy which plays these optimal actions from each configuration. Using Proposition 14, we can show that such a policy is indeed optimal. ◀

Notice that the above proofs crucially rely on the fact that the discount $\gamma < 1$. Indeed, for $\gamma = 1$ (infinite horizon reachability *without* discount), playing positionally the optimal action may lead to very bad outcome. E.g., in Figure 1, the optimal action from $(\ell_0, x < 1)$ is to play action b , looping forever in location ℓ_0 , as the expected reward associated with playing b is higher (the supremal value is 1) than the reward from playing a which has some risk associated with it (of value $1 - \varepsilon$). When there is a discount $\gamma < 1$, playing a becomes optimal from $(\ell_0, x < 1)$ for x close to 1, as waiting for a better value of x is not worth it.

5 Computing ε -optimal Policies

5.1 Mode Policies for DSTA with 1 clock

We now restrict ourselves to DSTA with a single clock x . 1-clock polytopes are intervals (called modes), i.e. they can be represented by two bounds in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$. We call *mode policies* such polytope policies. A mode policy σ of DSTA \mathcal{A} with control state \mathcal{L} is represented in the following finite way: $\sigma = (\mathcal{M}_{\sigma}^{\ell}, \sigma^{\ell})_{\ell \in \mathcal{L}}$ with:

- $\mathcal{M}_{\sigma}^{\ell}$ is a partition of $\mathbb{R}_{\geq 0}$ in a finite set of modes $M_1 \dots M_k \in \mathcal{M}_{\sigma}^{\ell}$ (the modes of ℓ),
- For all $\ell \in \mathcal{L}$, we have $\sigma^{\ell} : \mathcal{M}_{\sigma}^{\ell} \rightarrow E$ gives the action associated with each mode of ℓ .

We will represent the bounds of the interval with rational numbers in \mathbb{Q} , wlog as we only need ε -approximations. Mode-policies allow for a finite number of points of discontinuity.

Closeness or openness of intervals is not a concern as there is probability 0 to reach the exact boundary *between two modes* in a control state. What happens at b for the maximal mode $[a, b]$ of a control state might be important though (e.g. with $b = a$ caused by an invariant), so we have one special mode at the end of the interval for $[b, b]$ whenever $b < +\infty$.

5.2 Symbolic Value Iteration for mode policies

We proceed in two steps: First, we provide our symbolic value iteration Algorithm 1 to define the exact mode policy σ_i^* obtained after i steps of the value iteration. Proposition 20 shows that the value Val_i^* of σ_i^* converges towards the optimal value, and given an $\varepsilon > 0$, it provides an effectively computable i such that $Val_i^* > Val^* - \frac{\varepsilon}{2}$. Lemma 19 proves that σ_i^* has a finite number of modes, for all $i \in \mathbb{N}$. We never compute the value function itself (we do not know how to represent it). Further, modes, e.g. the exact value of the (possibly non-rational) boundaries, cannot be computed perfectly.

■ **Algorithm 1** Symbolic Value Iteration Algorithm to define the exact mode policy σ_i^* .

```

1 Init:  $i := 0$  and  $\forall \ell \in \mathcal{L}, x \in \mathbb{R}_{\geq 0}, \iota \in \{\alpha, \beta\}$ , let  $\text{Val}_0^*(\ell, x) := \text{Val}_0^*(\ell, x, \iota) = 0$ ;
2  $\sigma_0^*$  has only one mode  $M = [0, \infty)$  for each  $\ell \in \mathcal{L}$  playing  $\sigma_0^{*\ell}(M) = \alpha$ ;
3 repeat
4   for each location  $\ell \in \mathcal{L}$  do
5     for each mode  $M \in \mathcal{M}_{\sigma_i}^\ell$  do
6       for each action  $\iota \in \{\alpha, \beta\}$ , let  $\ell'$  the destination of the transition  $\iota$  and
           $r = 1$  if the clock is reset,  $r = 0$  otherwise do
7         Define  $\text{Val}_{i+1}(\ell, x, \iota) := \mathcal{R}(\iota) + \gamma \text{IVal}_i^*(\ell', (1-r)x)$  for any  $x$ ;
8       end
9       Compute a minimal finite set  $\mathcal{M}_M^\ell$  of modes such that on each mode
           $\text{Val}_{i+1}(\ell, v, \alpha) - \text{Val}_{i+1}(\ell, v, \beta)$  does not change sign;
10      end
11      Set  $\mathcal{M}_{i+1}^\ell := \bigcup_{M \in \mathcal{M}_{\sigma_i}^\ell} \mathcal{M}_M^\ell$ 
12    end
13    Assign  $\sigma_{i+1}^* := (\mathcal{M}_{i+1}^\ell, \sigma_{i+1}^\ell)_{\ell \in \mathcal{L}}$  where for each control state  $\ell \in \mathcal{L}$  and mode
           $M \in \mathcal{M}_{i+1}^\ell$ , the policy is  $\sigma_{i+1}^{*\ell}(M) = \alpha$  if  $\text{Val}_{i+1}(\ell, x, \alpha) - \text{Val}_{i+1}(\ell, x, \beta) \geq 0$ 
          for  $x \in M$ , and  $\sigma_{i+1}^{*\ell}(M) = \beta$  otherwise;
14    Define  $\text{Val}_{i+1}^*(\ell, v) := \text{Val}_{i+1}(\ell, v, \sigma_{i+1}^\ell(M))$  for  $v \in M$ ;
15    Let  $i := i + 1$ ;
16 end

```

The second step handles these issues, by computing effectively a mode policy $\bar{\sigma}_1$ approximating σ_1^* , such that its value is at least $\overline{\text{Val}}_1 > \text{Val}_1^* - \varepsilon_1$. We set ε_1 sufficiently small such that inductively, we obtain a policy $\bar{\sigma}_i$ with value $\overline{\text{Val}}_i > \text{Val}_i^* - \frac{\varepsilon}{2}$. Overall, $\overline{\text{Val}}_i > \text{Val}_i^* - \varepsilon$.

To ease the presentation, we assume that only two actions $\{\alpha, \beta\}$ are available. It is easy to extend to more actions by comparing actions pairwise. We focus on modes where both α, β are allowed, as the policy is trivial if only one action is allowed. On these modes, Val is continuous. Recall that for $\text{I}(\ell) = [a, b]$, we denote $\text{IVal}(\ell, x) = \int_x^b \text{Val}(\ell, t) d\mu(\ell, t)$.

Step 1: Symbolic Function Analysis to define the exact mode policy.

The structure of Algorithm 1 is as follows: from initial value $\text{Val}_0 = 0$, we inductively define the next action value using Bellman Optimality (line 7). Then, the modes of this new value function are defined (line 9) and for each mode the best action is chosen (line 13). Finally the new value function is *defined* for each mode (line 14). Notice that we only need to *determine* the boundaries where the sign of $\text{Val}_i(\ell, x, \alpha) - \text{Val}_i(\ell, x, \beta)$ changes, we do not need to *compute* or *represent* the value function (*symbolic* nature). The algorithm relies on the crucial fact that the number of such boundaries at each step is finite (Lemma 19).

► **Lemma 19.** *Let $\ell \in \mathcal{L}$ a control state. For all $i \in \mathbb{N}$, defining $f : x \mapsto \text{Val}_i(\ell, x, \alpha) - \text{Val}_i(\ell, x, \beta)$, there is a $m \in \mathbb{N}$ such that for all $j \leq m$, there exists a sequence $(\tilde{f}^{(j)})_{j=0}^m$ of derivable functions, called the pseudo-derivatives of f , with:*

- the sign of $\tilde{f}^{(0)}$ and of f coincide in all points of M ,
- for $1 \leq j \leq m$, the sign of $\tilde{f}^{(j)}$ and of $\frac{d\tilde{f}^{(j-1)}}{dt}$ coincide in all points of M , and
- function $\tilde{f}^{(m)}$ has a constant sign over M .

In particular, functions $f, \tilde{f}^{(0)}$ change sign at most m times.

Line 13 computes the sign of $\text{Val}_{i+1}(\ell, x, \alpha) - \text{Val}_{i+1}(\ell, x, \beta) = f(x)$. As f changes sign at most m times (Lemma 19), we have at most $m + 1$ modes. The proof of Lemma 19 can be found in Section 5.4 and relies on the symbolic derivation of $\tilde{f}^{(j)}$ depending on the probability distributions $d\mu(\ell, t)$. We can now prove that Algorithm 1 produces an ε -policy:

► **Proposition 20.** *For all $\varepsilon > 0$, Algorithm 1 produces a mode-policy σ_ε^* that is ε -optimal. Further, one can effectively compute a step i after which $(\text{Val}^* - \varepsilon) < \text{Val}_{\sigma_i^*} \leq \text{Val}^*$.*

The proof of Proposition 20 can be found in Section 5.5. It is done in three steps: first show that Val_i^* converges, towards a value that is solution of the Bellman Optimality Equation. The third part of the proof provides the number of steps i to ensure a given ε .

To prove Theorem 9, we make symbolic Algorithm 1 *effective*, computing a mode policy $\overline{\sigma}_i^*$ approximating σ_i^* . In particular, we effectively approximate constants coming from integral computations in $(\tilde{f}^{(j)})_{j=0}^m$, then approximate their roots, to obtain approximated roots of f , giving \mathbb{Q} -boundaries arbitrarily close to the boundaries of modes of the symbolic σ_i^* .

Step 2: Numerical Analysis to effectively compute an approximated mode policy $\overline{\sigma}_i^*$.

Line 13 of Algorithm 1 asks to find the change of sign (or equivalently the roots) for the function f defined in Lemma 19. While the values of the roots may not be easily computable symbolically, we show in the following Lemma that we can approximate them with arbitrarily high precision using numerical analysis (e.g. using Ridder's method [Rid79]):

► **Lemma 21.** *Let f and m be as defined in Lemma 19. Then for all $\varepsilon_0 > 0$, one can compute $b_0 \leq a_1 \leq b_1 \leq \dots \leq a_m$ such that for all $j \leq m$, f does not change sign on $[b_j, a_{j+1}]$, and $|f(t)| \leq \varepsilon_1 = 5\varepsilon_0 \text{MaxValue}$ over all $t \in [a_j, b_j]$ and $|b_j - a_j| \leq \varepsilon_0$.*

The proof of Lemma 21 can be found in appendix. It is crucial that the number of intervals m does not depend upon the bounds ε_0 to obtain an effectively computable mode policy. From a value function Val , associated function f , and $b_0 \leq a_1 \leq b_1 \leq \dots \leq a_m$ from Lemma 21 using a small enough $\varepsilon_0 > 0$, we define the approximated mode policy $\overline{\sigma}$ as follows:

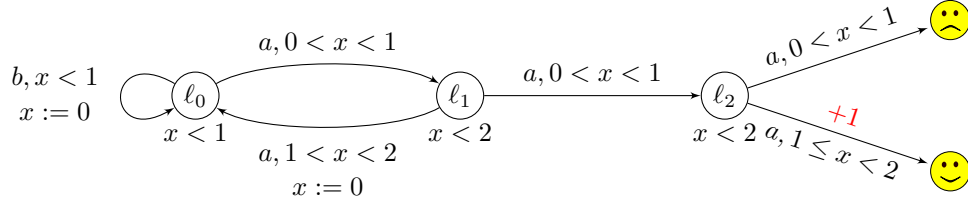
- Between (b_i, b_{i+1}) , the policy $\overline{\sigma}$ plays α if the sign of $f(\frac{b_i+a_{i+1}}{2})$ is > 0 . Indeed, between (b_i, a_{i+1}) , we know that playing α brings a higher Val_{i+1} than playing β . Between (a_{i+1}, b_{i+1}) , the difference is small so it does not matter much the choice α or β .
- Otherwise, $\overline{\sigma}$ plays β .

We denote $\overline{\text{Val}}(\ell, v) = \mathcal{R}(\overline{\sigma}(\ell, v)) + \gamma \text{IVal}(\ell', v_{r \leftarrow 0})$ the value function associated with $\overline{\sigma}$, for ℓ' reached by playing $\overline{\sigma}$ resetting r . Let Val_{+1} be the value function we would obtain by applying one iteration of Algorithm 1 from Val . As corollary of Lemma 21, we obtain:

► **Corollary 22.** $|\overline{\text{Val}}(\ell, x) - \text{Val}_{+1}(\ell, x)| < \varepsilon_1 = 5\varepsilon_0 \text{MaxValue}$.

We inductively compute $\overline{\text{Val}}_i, \overline{\sigma}_i$ from the previously computed $\overline{\text{Val}}_{i-1}, \overline{\sigma}_{i-1}$.

The proof of Theorem 9, provided in Section 5.6, shows that we can limit the effect of those successive approximations on the value of the resulting policy. We obtain $\overline{\text{Val}}_i > \text{Val}_i - \frac{\varepsilon}{2}$ by applying Corollary 22 repetitively for ε_0 chosen small enough. We conclude by combining this error for the i provided by Proposition 20, and using the fact that $\text{Val}(\overline{\sigma}_i) \geq \overline{\text{Val}}_i$.



5.3 Running example: methodology and numerical evaluation

We now roll out our procedure on an example, and compare with discretization [CHK+11].

Let us consider the DSTA \mathcal{A} from Figure 1. For convenience, we copy the picture hereafter. At initialization, there is a unique mode $[0, \infty)$ in all control state, choosing action a .

After one step, only ℓ_2 has a non-null value. The value of ℓ_2 is $\text{Val}_1(\ell_2, x) = \text{Val}^*(\ell_2, x) = 1$ if $x > 1$ and 0 if $x < 1$, as the only transition with non-null reward is the one from ℓ_2 to 😊. The value remains unchanged by later iterations. As there is no choice of action (only a available), ℓ_2 has a unique mode choosing a . It is not always possible to compute a closed-form solution for the value, and often one needs to keep the integral expression.

After two steps, the value in ℓ_1 is updated as $\text{Val}_1(\ell_2, x) > 0$ for some x . The value is $\text{Val}_2(\ell_1, x) = \text{Val}_3(\ell_1, x) = 0$ for $x > 1$ and $\frac{\gamma}{2-x} \int_{t \in [x, 1]} 1 dt = \gamma \frac{1-x}{2-x}$ for $x < 1$, where again we can give a closed form solution. This will be updated at later steps (≥ 4) after ℓ_0 gets a non-zero value and transition from ℓ_1 to ℓ_0 can bring some contribution to the reward as well. As there is no choice of action in ℓ_1 , there is a unique mode, choosing action a .

After three steps, the values of actions in ℓ_0 are updated for the first time. We have $\text{Val}_3(\ell_0, x, b) = 0$ and $\text{Val}_3(\ell_0, x, a) = \frac{\gamma}{2-x} \int_{t \in [x, 1]} \text{Val}_2(\ell_1, t) dt = \frac{\gamma^2}{2-x} \int_{t \in [x, 1]} \frac{1-t}{2-t} dt > 0$. We can notice that $\frac{1-t}{2-t} = 1 - \frac{1}{2-t}$. As $\int_{t \in [x, 1]} \frac{1}{2-t} dt = \int_{t \in (1, 2-x]} \frac{1}{t} dt = \log(2-x)$, this gives $\text{Val}_3(\ell_0, x, a) = \gamma^2 \frac{1-x - \log(2-x)}{2-x}$. So there is a unique mode, $[0, \infty)$ choosing action a .

The fourth steps for ℓ_0 is the first interesting case. To take its decision, value iteration considers $\text{Val}_3(\ell_0, x) = \gamma^2 \frac{1-x - \log(2-x)}{2-x}$ and $\text{Val}_3(\ell_1, x) = \gamma \frac{1-x}{2-x}$. We have:

$$\text{Val}_4(\ell_0, x, b) = \gamma^3 \int_{t \in [0, 1]} \frac{1-t - \log(2-t)}{2-t} dt; \quad \text{Val}_4(\ell_0, x, a) = \frac{\gamma^2}{1-x} \int_{t \in [x, 1]} \frac{1-t}{2-t} dt$$

To know whether one needs to choose action a or b in ℓ_0 at the 4th step of value iteration, one has to study the sign of $f = \text{Val}_4(\ell_0, x, b) - \text{Val}_4(\ell_0, x, a)$. When it is positive, the policy should play b , and a otherwise. The value $\text{Val}_4(\ell_0, x, b)$ is a constant C_1 not depending upon x , whose value can be approximated to $0.0666\gamma^3$. We *do not compute* a closed form solution for $\text{Val}_4(\ell_0, x, a)$ - this is not necessary and often not possible anyway.

We have $f(x) = C_1 - \frac{\gamma^2}{1-x} \int_{t \in [x, 1]} \frac{1-t}{2-t} dt$. We multiply f by $(1-x)$ which does not change its sign (as $x < 1$), defining $\tilde{f}^{(0)}(x) = f(x)(1-x)$. We compute $\frac{d\tilde{f}^{(0)}(x)}{dx} = \gamma^2 \frac{1-x}{2-x} - C_1$, getting rid of the integral. Multiplying by $(2-x)$, which is always positive, we define $\tilde{f}^{(1)}(x) = \gamma^2(1-x) - C_1(2-x)$. This is an affine function, which changes sign at most once, and whose root can be easily decided, depending on the fix value of γ . The function $\tilde{f}^{(1)}$ cancels at the same root, hence $\tilde{f}^{(0)}$, changes at most twice its sign, that is $f(x)$ changes at most twice its sign (3 modes maximum) at the same (at most 2) roots. Actually, for $\gamma = 0.95$, it has only 1 root r in $[0, 1)$, around 0.49863, so 2 modes:

$$M_4^{\ell_0}: \begin{array}{c} b \qquad a \\ \left[\text{-----} \left[\text{-----} \right) \right. \\ 0 \quad r \approx 0.49863 \quad 1 \end{array}$$

We finish the example by explaining how modes are taken into account inductively:

$$\text{Val}_5(\ell_0, x, b) = \gamma \int_{t \in (0, r)} \text{Val}_4(\ell_0, t, b) dt + \gamma \int_{t \in (r, 1)} \text{Val}_4(\ell_0, t, a) dt,$$

while $\text{Val}_5(\ell_0, x, a) = \text{Val}_4(\ell_0, x, a)$ as $\text{Val}_4(\ell_1, x, a) = 0$ for $x > 1$. By definition, $\text{Val}_5(\ell_0, x, a)$ does not depend upon x , it is a constant C_2 larger than $C_1 = \text{Val}_4(\ell_0, x, a)$ for $\gamma = 0.95$, which gives still 2 modes for $M_5^{\ell_0}$ with a switch at $r^5 > r$. For all i , $\text{Val}_i(\ell_1, x, a)$ is a constant for $x > 1$, while $\text{Val}_i(\ell_1, x, a) = \text{Val}_2(\ell_1, x, a)$ for $x < 1$. Hence, all subsequent value iteration for ℓ_0 have two modes, playing b on $(0, r^i)$ and a on $(r^i, 1)$, with r^i approaching r^* for the optimal strategy playing b on $(0, r^*)$ and a on $(r^*, 1)$.

Lemma 19 generalizes this process proving that there is only a finite number of sign changes for f , while Lemma 21 shows that the bounds of the intervals of the different roots/modes of $\tilde{f}^{(j)}$ can be approximated arbitrarily close (using Ridder's method [Rid79]).

Evaluation on the running example and comparison with time-discretization

We now evaluate the different constants needed on the running example. Consider a standard discount $\gamma = 0.95$. MaxReward= 1 gives (proposition 20) a number of value iterations of $i < 200$ for a precision of $\varepsilon = 10^{-5}$, and of $i < 300$ for $\varepsilon = 10^{-7}$. For each iteration i , the two constants $C_1^i = \text{Val}_i(\ell_0, x, b)$ independent of the choice of $x < 1$ and $C_2^i = \text{Val}_i(\ell_1, x, a)$ independent of the choice of $x > 1$ are evaluated, i.e. 2 evaluations of function at one given point x , so overall i , 400 or 600 evaluations of function at one point. Then, the roots root_i^1 of $\tilde{f}^{(1)}$ are computing using C_1^i, C_2^i , then the root of $\tilde{f}^{(0)}$ in $(0, \text{root}_i^1)$ if any, and the root in $(\text{root}_i^1, 1)$ if any, using Ridder's method. This yields 600, 900 call to Ridder's altogether. Each call needs respectively $\log(2 \cdot 10^5) < 13$ and $\log(2 \cdot 10^7) < 17$ iterations to ensure the $\varepsilon = \frac{1}{2}10^{-5}$ and $\varepsilon = \frac{1}{2}10^{-7}$ precision respectively. Each iteration of Ridder's method evaluates $f(x)$ at one given point x to narrow down where the root is (which side of $f(x)$ it is). Applying Corollary 22, one obtains a precision of 10^{-4} with 8200 evaluations of functions at one point, and 10^{-6} after 15900 evaluations of function at one point respectively.

Consider the time-discretization method of [CHK+11]. It cannot handle uniform distribution as used in ℓ_0, ℓ_1 , but for the sake of comparison, consider a 3 states system using only exponential distributions, and a single clock. Quoting the long technical report p22 http://www.cs.ox.ac.uk/people/taolue.chen/pub-papers/cdc_full.pdf of [CHK+11], the authors need to solve a >60000 states MDP for 10^{-4} precision, and > 1 million states MDP for 10^{-6} precision. Using Linear Programming, this would amount to the same number of variables, with a process far from linear. Using value iteration would be possible, but incur another error which would need to be taken into account, amounting to a similar 200 iterations times 60000 states for the 10^{-4} precision. An advantage is that the time-discretization also holds for more than one clock, although the complexity would be prohibitive, with an MDP with 1.2 billion states already for 2 clocks and the precision 10^{-4} , making it a moot advantage, on top of the bounds not applying for DSTAs.

5.4 Proof of Lemma 19

Notice that if we could compute a closed-form solution of the value functions after n steps, it would be easy to deduce the values of x where the sign of $f(x)$ changes. The problem is that we do not know closed-form solutions of integrals of more and more complicated functions.

Instead, we cleverly replace the integrals by a simpler functions $\tilde{f}^{(j)}$ which changes sign at the same value, derive, until we reach a polynomial where the lemma is trivially true, and analyze the roots of function $\tilde{f}^{(j)}$ by a backward induction. This does not work with ≥ 2 clocks as for functions of ≥ 2 variables, sign changes are not related with roots of derivatives.

We fix a location ℓ at iteration i . We present the proof in the case where neither playing α nor β resets x (other cases are simpler) and all distributions on delays are uniform (which is more complex than exponentials). Other cases can be found in appendix. For $\iota \in \{\alpha, \beta\}$, we denote $(M_j = (x_j^\iota, x_{j+1}^\iota))_{0 \leq j \leq J_\iota}$ the modes of σ_i in ℓ_ι that can be reached from x . We have:

$$\text{Val}_{i+1}(\ell, x, \iota) = \mathcal{R}(\iota) + \gamma \frac{1}{x_{J_\iota+1}^\iota - x} \left(\int_{x \leq t \leq x_1^\iota} \text{Val}_i(\ell_\iota, t, \sigma_i^{\ell_\iota}(M_0)) dt + \sum_{j=1}^{J_\iota} \int_{x_j^\iota \leq t \leq x_{j+1}^\iota} \text{Val}_i(\ell_\iota, t, \sigma_i^{\ell_\iota}(M_j)) dt \right)$$

Let $x_m = \min(x_0^\alpha, x_0^\beta)$. There exists constants C_ι , C'_ι and C''_ι such that:

$$\text{Val}_{i+1}(\ell, x, \iota) = C_\iota + C'_\iota \frac{1}{x_{J_\iota+1}^\iota - x} \int_{x \leq t < x_m} \text{Val}_i(\ell_\iota, t, \sigma_i^{\ell_\iota}(M_0)) dt + C''_\iota \frac{1}{x_{J_\iota+1}^\iota - x}.$$

Lemma 19 only needs symbolic functional analysis (value of m and existence of functions $(\tilde{f}^{(j)})_{j \leq m}$), so we do not need in Lemma 19 the actual value of C_ι 's, only their existence. Later, in Lemma 21, we will however need to approximate these constants. This is done through numerical integration (Newton-Cotes and Generalized Gauss-Laguerre quadrature). To study the sign of $f_{i+1}(x) = \text{Val}_{i+1}(\ell, x, \alpha) - \text{Val}_{i+1}(\ell, x, \beta)$ we study the sign of $\tilde{f}^{(0)} = (x_{J_\alpha+1}^\alpha - x)(x_{J_\beta+1}^\beta - x)f_{i+1}(x)$, which is the same. For C a constant, we write $\tilde{f}^{(0)}$ as:

$$\begin{aligned} \tilde{f}^{(0)}(x) = & C(x_{J_\alpha+1}^\alpha - x)(x_{J_\beta+1}^\beta - x) \int_{x \leq t \leq x_m} \text{Val}_i(\ell_\alpha, t) dt + C_\alpha(x_{J_\beta+1}^\beta - x) \\ & + C'_\beta(x_{J_\alpha+1}^\alpha - x) \int_{x \leq t \leq x_m} \text{Val}_i(\ell_\beta, t) dt + C_\beta(x_{J_\alpha+1}^\alpha - x) \end{aligned}$$

We derive $\tilde{f}^{(0)}$ once to obtain $\tilde{f}^{(1)}$, then once more obtaining for some C_1, C_2, C_3, C_4, C_5 :

$$g(x) = C_1 + C_2 \text{Val}_i(\ell_\alpha, x) + C_3(x_{J_\beta+1}^\beta - x) \text{Val}'_i(\ell_\alpha, x) + C_4 \text{Val}_i(\ell_\beta, x) + C_5(x_{J_\alpha+1}^\alpha - x) \text{Val}'_i(\ell_\beta, x)$$

There is no more integral terms in $g^{(2)}(x)$. Function $g^{(2)}(x)$ is of the generic form:

$$G_i(x) = P(x) + \sum_{k \leq K} P_k^\alpha(x) \text{Val}_i^{(k)}(\ell_\alpha, x) + \sum_{k \leq K} P_k^\beta(x) \text{Val}_i^{(k)}(\ell_\beta, x),$$

for some polynomials $P, P_0^\alpha, \dots, P_K^\alpha, P_0^\beta, \dots, P_K^\beta$. Generalizing the method above, we show in appendix by induction over $i \in \mathbb{N}$, that there exists a $m' \in \mathbb{N}$ such that for all $j \leq m'$, there exists a sequence $(\tilde{g}^{(j)})_{j=0}^{m'}$ of derivable functions with:

- the sign of $\tilde{g}^{(0)}$ and of $G_i(x)$ coincide in all points of M ,

- for $3 \leq j \leq m'$, the sign of $\tilde{g}^{(j)}$ and of $\frac{d\tilde{g}^{(j-1)}}{dt}$ coincide in all points of M , and
- function $\tilde{g}^{(m')}$ has a constant sign over M .

Applying this result, it suffices to choose $m = m' + 2$ and $\tilde{f}^{(j)} = \tilde{g}^{(j-2)}$ for all $2 \leq j \leq m' + 2$.

5.5 Proof of Proposition 20

First, we define $\text{MaxReward} = \max_{(\ell, \iota) \in \mathcal{L} \times E} (\mathcal{R}(\iota))$, the maximal reward over all the transitions, as well as $\text{MaxValue} = \frac{\text{MaxReward}}{1-\gamma}$ an upper bound on values in all the states.

We now show that the value iteration process indeed converges towards the optimal value $\text{Val}^*(\mathcal{A})$. For this, we first demonstrate that $(\text{Val}_i)_{i \in \mathbb{N}}$ converges as it is non-decreasing and upper bounded. Then we prove that the value it converges to is a (bounded) solution to Bellman Optimality Equations, which allows us to stop the algorithm after a computable number i of steps, with the guarantee that the policy is ε -optimal.

Proof. Value Iteration converges. Recall that $\text{Val}_0(\ell, v) = 0$ for all ℓ, v , and that all rewards are non negative. By an induction on i , we show that the Val_i have an upper bound:

$$\forall i \in \mathbb{N}, \forall (\ell, v) \in \mathcal{L} \times \mathbb{R}_{\geq 0}, \text{Val}_i(\ell, v) \leq \frac{\text{MaxReward}}{1-\gamma} = \text{MaxValue}$$

Another induction on i shows that the v_i are increasing: for all $i \geq 0$, $\text{Val}_{i+1} \geq \text{Val}_i$.

Initialization: $\text{Val}_0 \leq \text{Val}_1$ is trivial as $\text{Val}_0 = 0$ and no reward is negative.

Now, assume $\text{Val}_{J+1} \geq \text{Val}_J$. Let $(\ell, v) \in \mathcal{L} \times \mathbb{R}_{\geq 0}$.

$$\begin{aligned} \text{Val}_{J+2}(\ell, v) &= \max_{\iota \in E} \mathcal{R}(\iota) + \gamma \int_{\delta \in I(\ell', v_{[r \leftarrow 0]})} \text{Val}_{J+1}(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})} \\ &\quad \text{applying the induction hypothesis, we get} \\ &\geq \max_{\iota \in E} \mathcal{R}(\iota) + \gamma \int_{\delta \in I(\ell', v_{[r \leftarrow 0]})} \text{Val}_J(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})} \\ &= \text{Val}_{J+1}(\ell, v) \end{aligned}$$

Thus (Val_i) is increasing with i , bounded, hence converges towards some Val_∞ .

Val_∞ is a solution to Bellman Optimality Equation. For all $(\ell, v) \in \mathcal{L} \times \mathbb{R}_{\geq 0}$ and i :

$$\text{Val}_{i+1}(\ell, v) = \max_{\iota \in \Sigma} \mathcal{R}(\iota) + \gamma \int_{\delta \in I(\ell', v_{[r \leftarrow 0]})} \text{Val}_i(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})}$$

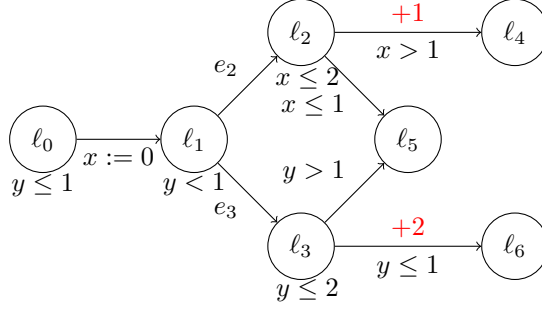
At the limit, we obtain:

$$\begin{aligned} \text{Val}_\infty(\ell, v) &= \lim_{i \rightarrow \infty} \max_{\iota \in \Sigma} \mathcal{R}(\iota) + \gamma \int_{\delta \in I(\ell', v_{[r \leftarrow 0]})} \text{Val}_i(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})} \\ &= \max_{\iota \in \Sigma} \mathcal{R}(\iota) + \gamma \lim_{i \rightarrow \infty} \int_{\delta \in I(\ell', v_{[r \leftarrow 0]})} \text{Val}_i(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})} \\ \text{Val}_\infty(\ell, v) &= \max_{\iota \in \Sigma} \mathcal{R}(\iota) + \gamma \int_{\delta \in I(\ell', v_{[r \leftarrow 0]})} \text{Val}_\infty(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})} \end{aligned}$$

Val_∞ is indeed a bounded solution of the Bellman Optimality Equation. Under the hypothesis that the optimal value Val^* exists, it is the only bounded solution of the Bellman Optimality Equation: we therefore have $\text{Val}_\infty = \text{Val}^*$.

As the mode-policies produced by Value Iteration are greedy over the successive Val_i that converge toward Val^* , their values converge toward Val^* .

Expression of the approximation as a function of i . As stated, σ_i^* plays optimally according to Val_i , thus $\text{Val}_{\sigma_i^*} \geq \text{Val}_i$. Furthermore, by unraveling the expression of Val_i up to $\text{Val}_0 = 0$ and comparing it to Val_∞ (using the Bellman Optimality Equation), we obtain: $\text{Val}_\infty(\ell, v) - \text{Val}_i(\ell, v) \leq \gamma^i \text{MaxValue}$. Given $\varepsilon > 0$, we set $i = \frac{\log(\varepsilon) - \log(1-\gamma) + \log(\text{MaxReward})}{\log(\gamma)}$, to obtain $\text{Val}_{\sigma_i^*} \geq \text{Val}_i \geq \text{Val}_\infty - \varepsilon = \text{Val}^* - \varepsilon$. ◀



■ **Figure 2** A simple DSTA with two clocks without loops

5.6 Proof of Theorem 9

Let $\varepsilon > 0$ be a small constant. Using Proposition 20 we know that we can compute i such that $\text{Val}_i^* \geq \text{Val}^* - \frac{\varepsilon}{2}$. In particular, $\text{Val}_i^*(\lambda) \geq \text{Val}^*(\lambda) - \frac{\varepsilon}{2}$ (where λ is the empty run). We choose approximation factor ε_0 as follows for Corollary 22 such that $\overline{\text{Val}}_i(\lambda) \geq \text{Val}_i^*(\lambda) - \frac{\varepsilon}{2}$.

We show by induction on j that $\overline{\text{Val}}_j \geq \text{Val}_j^* - \sum_{k=0}^{j-1} \gamma^k \times (5\varepsilon_0 \text{MaxValue})$ with ε_0 as defined in Lemma 21. For $j = 0$ we have $\overline{\text{Val}}_0 = \text{Val}_0$. Inductive step: suppose the result for a given j . Then, by Corollary 22 and Bellman Optimality Equation:

$$\overline{\text{Val}}_{j+1}(\ell, v) \geq \max_{\iota} (\mathcal{R}(\iota) + \gamma \cdot \overline{\text{IV}}\text{al}_j(\text{loc}'_{\iota}, v_{[r_{\iota}, \leftarrow 0]})) - 5\varepsilon_0 \text{MaxValue}$$

with r_{ι} the reset associated with ι in $(\ell, v + \delta)$ and ℓ'_{ι} the target location. Using the induction hypothesis we then have

$$\overline{\text{Val}}_{j+1}(\ell, v) \geq \max_{\iota} (\mathcal{R}(\iota) + \gamma \cdot \overline{\text{IV}}\text{al}_j(\text{loc}'_{\iota}, v_{[r_{\iota}, \leftarrow 0]})) - 5 \text{MaxValue} \left(\gamma \times \sum_{k=0}^{j-1} \gamma^k \varepsilon_0 - \varepsilon_0 \right).$$

By definition of Val_{j+1} we then have our result which finished the proof by induction.

We thus fix $\frac{\varepsilon}{2} \geq \sum_{k=0}^{i-1} \gamma^k 5\varepsilon_0 \text{MaxValue}$ i.e. $\varepsilon_0 \leq \frac{\varepsilon}{10 \text{MaxValue}} \times \frac{(1-\gamma)}{(1-\gamma^i)}$, which we can effectively compute. This ends the proof of Theorem 9.

5.7 Proofs of Theorem 10 : Counter example with 2 clocks

To prove Theorem 10, we exhibit a simple DSTA with two clocks and without loops (Figure 2) in which the optimal value can not be represented using polytopes.

The only choice is between e_2 and e_3 in ℓ_1 . The optimal values in ℓ_4 , ℓ_5 and ℓ_6 are simply 0 and the optimal values in ℓ_2 and ℓ_3 can be correctly computed in one step as:

$\text{Val}^*(\ell_2, (x, y)) = 1$ if $x > 1$ and 0 otherwise and $\text{Val}^*(\ell_3, (x, y)) = 2$ if $y \leq 1$ and 0 otherwise.

Thus the values of e_2 and e_3 are correctly computed in two steps as:

$$\text{Val}^*(\ell_1, (x, y), e_2) = \frac{1}{2-x} \int_{\delta \in]1-x, 2-x]} 1 dt = \frac{1}{2-x}$$

$$\text{Val}^*(\ell_1, (x, y), e_3) = \frac{2}{2-y} \int_{\delta \in]0, 1-y]} 1 dt = 2 \times \frac{1-y}{2-y}$$

The optimal value in ℓ_1 is thus maximum of these two values. It comes that e_2 is played when $\frac{1}{2-x} - 2 \times \frac{1-y}{2-y} > 0$ i.e. $3y + 2x - 2xy - 2 > 0$ which cannot be represented by a polytope.

6 Conclusion

Over the span of this paper, we defined a framework for Reinforcement Planning in dense time with discontinuities, modelled by DSTAs. In the presence of a single clock, we developed an algorithm that computes a ε -optimal policy over an uncountable number of configurations in bounded time, represented in a finite way as a mode policy. Notice that we do not use a closed-form solution for the value function itself: we believe such closed-form solutions cannot be computed in general even for 1-clock DSTA, as some integrals do not have closed-form solutions. Further, we did not rely on a discretization of time in ε -steps, in order to keep the number of mode minimal which we showed to be efficient on our running example w.r.t. time-discretization. This made the proofs particularly involved.

An interesting question is what happens if we use policy iteration instead of value iteration. This technique considers the reward associated with a policy, and thus demands more complex computations than value iteration which only considers bounded paths. We do not know how to evaluate them as we do not have closed-form solution of the value function.

Last, we supposed that we knew all the parameters of our model (rewards, distributions on delays) as it is the case in RP. In untimed settings, Reinforcement Learning developed techniques to work on partially unknown models by exploring them while trying to limit the necessary sample size. A interesting extension of this work would be to study a possible adaptation of exploration algorithms in dense time, which however reveals to be very challenging when dealing with Timed automata and its extensions [Gri08, HJM20].

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A Proofs of Section 4

A.1 Proof of Section 4.1

A.1.1 Proof of Property 11

Proof. We make the proof for the configuration-value. The action-value follows the same ideas.

A.1.1.1 $\text{Val}_\sigma(\rho)$ is solution to the Bellman Equation

we show this by unfolding the definition.

$$\begin{aligned} \text{Val}_\sigma(\rho) &= \lim_{n \rightarrow \infty} \sum_{(e_i)_{1 \leq i \leq n}} (P((e_i)_i | \rho) \times \mathcal{R}((e_i)_i)) \\ &= \lim_{n \rightarrow \infty} \sum_{(e_i)_{1 \leq i \leq n}} (\sigma(\rho)(e_1) \int_{\delta \in I(\ell_1, v')} P_\sigma((e_i)_{i \geq 2} | \rho \cdot (e_1, \delta)) d\mu_{(\ell_1, v')} \times (\mathcal{R}(e_1) + \gamma \mathcal{R}((e_i)_{i \geq 2}))) \end{aligned}$$

with ℓ_1 the target location of e_1 , r_1 its reset and $v' = v_{[r_1 \leftarrow 0]}$. We can separate the reward according to what depends on e_1 and what depends on $(e_i)_{i \geq 2}$. Notice that $\sum_{(e_i)_{2 \leq i \leq n}} \int_{\delta \in I(\ell_1, v')} P_\sigma((e_i)_{i \geq 2} | \rho \cdot (e_1, \delta)) d\mu_{(\ell_1, v')} = 1$ as the sum and integral can be inverted.

$$\begin{aligned} \text{Val}_\sigma(\rho) &= \lim_{n \rightarrow \infty} \sum_{e_1} \sigma(\rho)(e_1) \left(\mathcal{R}(e_1) + \right. \\ &\quad \left. \sum_{(e_i)_{2 \leq i \leq n}} \int_{\delta \in I(\ell_1, v')} P_\sigma((e_i)_{i \geq 2} | \rho \cdot (e_1, \delta)) d\mu_{(\ell_1, v')} \times \gamma \mathcal{R}((e_i)_{i \geq 2}) \right) \end{aligned}$$

Now, the first part of the sum on e_1 does not depend on n and can be taken out of the limit. The sum on $(e_i)_{2 \leq i \leq n}$ can be put inside the integral as it is a finite sum. Finally, as the limit of the expression for each e_1 exist and the values for n steps converge uniformly towards the value, the limit can be put inside the sum on e_1 and the integral, which yields:

$$\begin{aligned} \text{Val}_\sigma(\rho) &= \sum_{e_1} \sigma(\rho)(e_1) \left(\mathcal{R}(e_1) + \right. \\ &\quad \left. \int_{\delta \in I(\ell_1, v')} \lim_{n \rightarrow \infty} \sum_{(e_i)_{2 \leq i \leq n}} P_\sigma((e_i)_{i \geq 2} | \rho \cdot (e_1, \delta)) d\mu_{(\ell_1, v')} \times \gamma \mathcal{R}((e_i)_{i \geq 2}) \right) \\ &= \sum_{e_1} \sigma(\rho)(e_1) \left(\mathcal{R}(e_1) + \int_{\delta \in I(\ell_1, v')} \text{Val}_\sigma(\rho \cdot (e_1, \delta)) d\mu_{(\ell_1, v')} \right). \end{aligned}$$

A.1.1.2 There can only be one bounded solution to the Bellman Equation

we suppose we have two bounded functions F, F' , both solution of the Bellman Equation for a given policy σ . Since F and F' are bounded, we can introduce $\mathcal{D} = \sup_{\rho \in \text{Runs}(\mathcal{A})} |F(\rho) - F'(\rho)|$.

Let ρ by an history ending in a configuration $(\ell, v) \in \mathcal{L} \times \mathcal{V}_C$ of the DSTA. We study:

$$|F(\rho) - F'(\rho)| = \sum_{e \in E} \sigma(\rho)(e) \times |\gamma| \int_{\delta \in I(\ell_1, v_{[r \leftarrow 0]})} (F(\rho.(e, \delta)) - F'(\rho.(e, \delta))) d\mu_{(\ell_1, v_{[r \leftarrow 0]})} |$$

with r the reset of e and ℓ_1 its target. We unfold n times the difference $F - F'$ and note ρ_i the histories with $\rho_0 = \rho$ to get:

$$|F(\rho) - F'(\rho)| = \sum_{(e_i)_{1 \leq i \leq n}} \prod_{i=1}^n \sigma(\rho_{i-1})(e_i) \times |\gamma|^n \iint (F(\rho_n) - F'(\rho_n)) d\mu |$$

The following comes from applying the bound \mathcal{D} :

$$|F(\rho) - F'(\rho)| \leq \sum_{(e_i)_{1 \leq i \leq n}} \prod_{i=1}^n \sigma(\rho_{i-1})(e_i) \times \gamma^n | \iint \mathcal{D} d\mu | \leq \gamma^n \times \mathcal{D} .$$

By passing to the limit, we obtain that $|F(\rho) - F'(\rho)| \leq 0$ since $\gamma < 1$, for all histories ρ .

We have $F = F'$, there is only one unique bounded solution to the Bellman Equation in dense time. ◀

A.1.2 Proof of Corollary 12

Proof. ■ The first equality comes from writing the Bellman equation for $\sum_e \sigma(\rho)(e) \cdot \text{Val}_\sigma(\rho, e)$. Notice that it is a function of ρ only (for σ fixed) and satisfies the Bellman equality equation for configuration values. As it is also bounded we have our result.

■ Using the first equality, the second one comes simply from rewriting $\sum_e \sigma(\rho)(e) \cdot \text{Val}_\sigma(\rho, e)$ as $\text{Val}_\sigma(\rho)$ in the Bellman equation for action-values. ◀

A.1.3 Proof of Proposition 13

Proof. It is clear that σ' is pure and positional. We now prove that $\text{Val}_{\sigma'} \geq \text{Val}_\sigma$. Consider an history ρ_0 ending in ℓ_0, v_0 .

$$\text{Val}_\sigma(\rho_0) = \sum_{e_1} \sigma(\rho_0)(e_1) \cdot \left(\mathcal{R}(e_1) + \gamma \cdot \int_{\delta_1 \in I(\ell_1, v'_0)} \text{Val}_\sigma(\rho_1) d\mu_{(\ell_1, v'_0)} \right)$$

with (ℓ_1, v'_0) the configuration after playing e_1 and $\rho_1 = \rho_0 \cdot (e_1, v'_0 + \delta_1)$. Clearly we have that

$$\text{Val}_\sigma(\rho_0) \leq \max_{e_1 \in E} \sup_{\rho'_0 \text{ ending in } (\ell_0, v_0)} \mathcal{R}(e_1) + \gamma \cdot \int_{\delta_1 \in I(\ell_1, v'_0)} \text{Val}_\sigma(\rho'_1) d\mu_{(\ell_1, v'_0)}$$

with $\rho'_1 = \rho'_0 \cdot (e, v'_0 + \delta_1)$. We can then develop $v_\pi(\rho'_1)$ in the same way and apply the same upper bound. After repeating this process n times, we have, noting ρ'_i the histories encountered during the process, (ℓ_i, v_i) the configurations and δ_i the delays:

$$\text{Val}_\sigma(\rho_0) \leq \text{Val}_{\sigma'}^n(\rho_0) + \gamma^n \sup_{\rho'_n \text{ ending in } (\ell_n, v_n)} \iint_{\delta_1, \dots, \delta_{n+1}} \text{Val}_\sigma(\rho'_{n+1}) d\mu \dots d\mu$$

using the definition of σ' as the argmax on actions and histories. As Val_σ is bounded (Lemma 6), this expression converges to $\text{Val}_{\sigma'}$ when n goes to infinity. We thus have that $\text{Val}_{\sigma'} \geq \text{Val}_\sigma$. ◀

A.1.4 Proof of Proposition 14

As previously, the idea of the proof is to unfold the equation for n steps and take the limit by using the boundedness to conclude.

Proof. Fix a configuration (ℓ_0, v_0) .

$$\begin{aligned} F(\ell_0, v_0) &\leq F(\ell_0, v_0, \sigma(\ell_0, v_0)) \\ &= (\mathcal{R}(\sigma(\ell_0, v_0)) + \gamma \cdot \int_{\delta_1 \in I(\ell_1, v'_0)} (F((\ell_1, v'_0 + \delta_1)) d\mu_{(\ell_1, v'_0)}) . \end{aligned}$$

By using the same process n times we get

$$F(\ell_0, v_0) \leq \text{Val}_\sigma^n(\ell_0, v_0) + \gamma^n \iint_{(\delta_i)} F(\ell_n, v_n) d\mu .$$

As F is bounded (Lemma 6, as F is a value function since it follows Bellman Equations) this expression converges to Val_σ when n goes to infinity. We thus have that $\text{Val}_\sigma \geq F$.

For the strict inequality, notice that if there exists (ℓ, v) such that $F(\ell, v, \sigma(\ell, v)) > F(\ell, v)$ then taking $(\ell_0, v_0) = (\ell, v)$ gives a witness that $\text{Val}_\sigma > F$. Conversely, if $F(\ell, v, \sigma(\ell, v)) = F(\ell, v)$ for all configurations then we can conduct the same discussion as above and obtain $\text{Val}_\sigma = F$. ◀

A.1.5 Proof of Proposition 15

Proof. We make the proof for the configuration value. The proof for action-value follows similar ideas.

Let us show that Val^* is a solution to the Bellman optimality equation in dense time. We note \mathcal{R}_0 the global return of a run, \mathcal{R}_1 the residual return after one step and r_0 the first return.

$$\begin{aligned} \text{Val}^*(\ell, v) &= \sup_\sigma \text{Val}_\sigma(\ell, v) \\ &= \sup_\sigma \lim_n \sum_{\pi \in E^n} P_\sigma(\pi | (\ell, v)) \mathcal{R}(\pi) \\ &= \max_e \sup_\sigma \lim_n \sum_{\pi \in E^{n-1}} P_\sigma(e \cdot \pi | (\ell, v)) (\mathcal{R}(e) + \gamma \mathcal{R}(\pi)) \\ &= \max_e \left(\mathcal{R}(e) + \gamma \times \left(\sup_\sigma \lim_n \sum_{\pi \in E^{n-1}} P_\sigma(e \cdot \pi | (\ell, v)) \text{Reward}(\pi) \right) \right) . \end{aligned}$$

The result can be obtained as for Bellman Equations above by expanding the expression of $P_\sigma(e \cdot \pi | (\ell, v))$ and inverting the integral with the sum and limit.

We now show that there is at most one bounded solution to the Bellman Optimality Equation. Suppose we have two such bounded solutions Val_1 and Val_2 . By induction, we prove that for all configurations (ℓ, v) , for all $n \in \mathbb{N}$,

$$|\text{Val}_1(\ell, v) - \text{Val}_2(\ell, v)| \leq \gamma^n \sup_{(\ell', v')} |\text{Val}_1(\ell', v') - \text{Val}_2(\ell', v')|$$

We detail the proof of the first step. Without loss of generality, suppose $\text{Val}_2(\ell, v) \leq \text{Val}_1(\ell, v)$. Then:

$$\begin{aligned} |\text{Val}_1(\ell, v) - \text{Val}_2(\ell, v)| &= \text{Val}_1(\ell, v) - \text{Val}_2(\ell, v) \\ &= \max_{e \in E} \left(\mathcal{R}(e) + \gamma \cdot \int_{\delta \in I(\ell', v_{[r \leftarrow 0]})} \text{Val}_1(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})} \right) \\ &\quad - \max_{e \in E} \left(\mathcal{R}(e) + \gamma \cdot \int_{\delta \in I(\ell', v_{[r \leftarrow 0]})} \text{Val}_2(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})} \right) \end{aligned}$$

There exists $e_1 \in E$ such that

$$\text{Val}_1(\ell, v) = \mathcal{R}(e_1) + \gamma \cdot \int_{\delta \in I(\ell'_1, v_{[r \leftarrow 0]})} \text{Val}_1(\ell'_1, v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell'_1, v_{[r \leftarrow 0]})}$$

with ℓ'_1 and r the corresponding target state and reward. Therefore:

$$\begin{aligned} |\text{Val}_1(\ell, v) - \text{Val}_2(\ell, v)| &= \mathcal{R}(e_1) + \gamma \cdot \int_{\delta \in I(\ell'_1, v_{[r \leftarrow 0]})} \text{Val}_1(\ell'_1, v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell'_1, v_{[r \leftarrow 0]})} \\ &\quad - \max_{e \in E} \left(\mathcal{R}(e) + \gamma \cdot \int_{\delta \in I(e', v_{[r \leftarrow 0]})} \text{Val}_2(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(e', v_{[r \leftarrow 0]})} \right) \\ &\leq \mathcal{R}(e_1) + \gamma \cdot \int_{\delta \in I(\ell'_1, v_{[r \leftarrow 0]})} \text{Val}_1(\ell'_1, v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell'_1, v_{[r \leftarrow 0]})} \\ &\quad - \mathcal{R}(e_1) + \gamma \cdot \int_{\delta \in I(\ell'_1, v_{[r \leftarrow 0]})} \text{Val}_2(\ell'_1, v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell'_1, v_{[r \leftarrow 0]})} \\ &\leq \gamma \cdot \int_{\delta \in I(\ell'_1, v_{[r \leftarrow 0]})} \text{Val}_1(\ell'_1, v_{[r \leftarrow 0]} + \delta) - \text{Val}_2(\ell'_1, v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell'_1, v_{[r \leftarrow 0]})} \\ &\leq \gamma \cdot \int_{\delta \in I(\ell'_1, v_{[r \leftarrow 0]})} \sup_{(\ell', v')} |\text{Val}_1(\ell', v') - \text{Val}_2(\ell', v')| d\mu_{(\ell'_1, v_{[r \leftarrow 0]})} \\ |\text{Val}_1(\ell, v) - \text{Val}_2(\ell, v)| &\leq \gamma \cdot \sup_{(\ell', v')} |\text{Val}_1(\ell', v') - \text{Val}_2(\ell', v')| \end{aligned}$$

The induction uses this result and has a similar idea. Since $\gamma < 1$, for all configurations (ℓ, t) we obtain that

$$|\text{Val}_1(\ell, v) - \text{Val}_2(\ell, v)| \leq 0$$

thus there can only be one bounded solution to the Bellman Optimality Equation in dense time. \blacktriangleleft

A.1.6 Proof of Corollary 16

Proof. The first equality can be deduced by writing $\max_e \text{Val}^*(\ell, v, e)$ and unfolding it using the Bellman optimality equation for action-value. We then find that it is a solution to the optimality equation for configuration value and by uniqueness have our result.

The second equality comes from writing the optimality equation for action value and then replacing $\max_{e \in E} \text{Val}^*(\ell', v_{[r \leftarrow 0]} + \delta, e)$ by $\text{Val}^*(\ell', v_{[r \leftarrow 0]} + \delta)$ using the first equality of this corollary. \blacktriangleleft

A.1.7 Proof of Corollary 17

Proof. As σ' takes the maximum of the returns according to σ from all possible actions, it is clear that $\text{Val}_\sigma(\ell, v, \sigma'(\ell, v)) \geq \text{Val}_\sigma(\ell, v)$. Hence we have $\text{Val}_{\sigma'} \geq \text{Val}_\sigma$ by Proposition 14.

Notice that

$$\text{Val}_{\sigma'} = \max_{e \in E} \left(\mathcal{R}(e) + \gamma \cdot \int_{\delta \in I(\ell', v_{[r \leftarrow 0]})} \text{Val}_\sigma(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})} \right)$$

by definition of σ' and that in the case $\text{Val}_{\sigma'} = \text{Val}_\sigma$, this rewrites as

$$\text{Val}_{\sigma'} = \max_{e \in E} \left(\mathcal{R}(e) + \gamma \cdot \int_{\delta \in I(\ell', v_{[r \leftarrow 0]})} \text{Val}_{\sigma'}(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})} \right)$$

which is exactly the Bellman Optimality Equation for configurations values. Hence we have that σ' is optimal (and so is σ). \blacktriangleleft

A.1.8 Proof of Theorem 18

Proof. By Corollary. 16 we have that Val^* satisfy the hypothesis of Proposition 14. Hence by considering the policy σ defined such that for all configurations

$$\sigma(\ell, v) = \arg \max_{e \in E} \left(\mathcal{R}(e) + \gamma \cdot \int_{\delta \in I(\ell', v_{[r \leftarrow 0]})} \text{Val}^*(\ell', v_{[r \leftarrow 0]} + \delta) d\mu_{(\ell', v_{[r \leftarrow 0]})} \right)$$

with r the reset of e and ℓ' its target, we have that $\text{Val}_\sigma = \text{Val}^*$. Indeed by definition of σ , $\text{Val}^*(\ell, v, \sigma(\ell, v)) \geq \text{Val}^*(\ell, v)$ and by optimality of Val^* , $\text{Val}^*(\ell, v, \sigma(\ell, v)) \leq \text{Val}^*(\ell, v)$. ◀

B Proofs of Section 5

B.1 Proof of Lemma 19

We give here a complete proof Lemma 19.

We fix a location ℓ and a step i . We will make the proof by discussing on the modes that can be reached in the target locations ℓ_α and ℓ_β reached by playing transition α and β .

First, assume that neither playing α nor β resets the clock value of our DSTA. We conduct this proof first as it is the most complex/general case, while resets only create constant values that are easier to handle.

Let $v \in M$ be a clock value with x the value of the unique clock and ι a transition in $\{\alpha, \beta\}$. Using the relation between $\text{Val}_{i+1}(\ell, v, \iota)$ and $\text{Val}_i(\ell_\iota, v')$ and the relation between action-value and configuration value, we obtain ¹:

$$\begin{aligned} \text{Val}_{i+1}(\ell, x, \iota) = & \mathcal{R}(\iota) + \gamma \int_{x \leq t \leq x_1^t} \text{Val}_i(\ell_\iota, t, \sigma_i^{\ell_\iota}(M_0)) d\mu_{(\ell_\iota, t)} \\ & + \gamma \sum_{j=1}^{J_\iota} \int_{x_j^t \leq t \leq x_{j+1}^t} \text{Val}_i(\ell_\iota, t, \sigma_i^{\ell_\iota}(M_j)) d\mu_{(\ell_\iota, t)} \end{aligned}$$

with $(M_j = (x_j^t, x_{j+1}^t))_{0 \leq j \leq J_\iota}$ the modes of σ_i in ℓ_ι that can be reached from x .

We now partition the proof in three cases, depending on the distributions μ encountered in ℓ_α and ℓ_β :

B.1.1 Case: All distributions on delays are uniform distributions

Then, we have for $\iota \in \{\alpha, \beta\}$:

$$\begin{aligned} \text{Val}_{i+1}(\ell, x, \iota) = & \mathcal{R}(\iota) + \gamma \frac{1}{x_{J_\iota+1}^t - x} \left(\int_{x \leq t \leq x_1^t} \text{Val}_i(\ell_\iota, t, \sigma_i^{\ell_\iota}(M_0)) dt \right. \\ & \left. + \sum_{j=1}^{J_\iota} \int_{x_j^t \leq t \leq x_{j+1}^t} \text{Val}_i(\ell_\iota, t, \sigma_i^{\ell_\iota}(M_j)) dt \right) \end{aligned}$$

¹ in the following we will exploit the fact that we have only 1 clock to simplify the expressions by replacing some delays by valuations of the unique clock.

We will want to discuss the difference of $\text{Val}_{i+1}(\ell, x, \alpha)$ and $\text{Val}_{i+1}(\ell, x, \beta)$. For this, fix $x_m = \min(x_0^\alpha, x_0^\beta)$. When considering $\text{Val}_{i+1}(\ell, x, \iota)$ as a function of x we can then simplify the expression as follows: there exists constants C_ι , C'_ι and C''_ι such that:

$$\text{Val}_{i+1}(\ell, x, \iota) = C_\iota + C'_\iota \frac{1}{x_{J_\iota+1}^\iota - x} \int_{x \leq t < x_m} \text{Val}_i(\ell_\iota, t, \sigma_i^{\ell_\iota}(M_0)) dt + C''_\iota \frac{1}{x_{J_\iota+1}^\iota - x}.$$

Notice that Lemma 19 only needs symbolic functional analysis (value of m and existence of functions $(\tilde{f}^{(j)})_{j \leq m}$, so we do not need in Lemma 19 the actual value of C_ι 's, only their existence. Later, in Lemma 21, we will however need to approximate these constants (and others defined hereafter). This is done through numerical integration (Newton-Cotes and Generalized Gauss–Laguerre quadrature).

To study the sign of $f_{i+1}(x) = \text{Val}_{i+1}(\ell, x, \alpha) - \text{Val}_{i+1}(\ell, x, \beta)$ we study the sign of $\tilde{f}^{(0)}(x) = (x_{J_\alpha+1}^\alpha - x)(x_{J_\beta+1}^\beta - x)f_{i+1}(x)$, which is the same. With C a constant, $\tilde{f}^{(0)}$ is of the form:

$$\begin{aligned} \tilde{f}^{(0)}(x) = & C(x_{J_\alpha+1}^\alpha - x)(x_{J_\beta+1}^\beta - x) + C'_\alpha(x_{J_\beta+1}^\beta - x) \int_{x \leq t \leq x_m} \text{Val}_i(\ell_\alpha, t) dt + C_\alpha(x_{J_\beta+1}^\beta - x) \\ & + C'_\beta(x_{J_\alpha+1}^\alpha - x) \int_{x \leq t \leq x_m} \text{Val}_i(\ell_\beta, t) dt + C_\beta(x_{J_\alpha+1}^\alpha - x) \end{aligned}$$

We derive $\tilde{f}^{(0)}$ twice to remove the integral, and obtain for some constants C_1, C_2, C_3, C_4, C_5 fuction:

$$g(x) = C_1 + C_2 \text{Val}_i(\ell_\alpha, x) + C_3(x_{J_\beta+1}^\beta - x) \text{Val}'_i(\ell_\alpha, x) + C_4 \text{Val}_i(\ell_\beta, x) + C_5(x_{J_\alpha+1}^\alpha - x) \text{Val}'_i(\ell_\beta, x)$$

Interestingly, there is no more integral terms in $g(x)$. Function $g(x)$ is thus of the generic form:

$$g(x) = P(x) + \sum_{k \leq K} P_k^\alpha(x) \text{Val}_i^{(k)}(\ell_\alpha, x) + \sum_{k \leq K} P_k^\beta(x) \text{Val}_i^{(k)}(\ell_\beta, x),$$

for some polynomials $P, P_0^\alpha, \dots, P_K^\alpha, P_0^\beta, \dots, P_K^\beta$. Generalizing the method above, we show by induction over $i \in \mathbb{N}$ that for every polynomials $P, P_0^\alpha, \dots, P_K^\alpha, P_0^\beta, \dots, P_K^\beta$, denoting

$$G_i(x) = P(x) + \sum_{k \leq K} P_k^\alpha(x) \text{Val}_i^{(k)}(\ell_\alpha, x) + \sum_{k \leq K} P_k^\beta(x) \text{Val}_i^{(k)}(\ell_\beta, x),$$

there exists a $m' \in \mathbb{N}$ such that for all $j \leq m'$, there exists a sequence $(\tilde{g}^{(j)})_{j=0}^{m'}$ of derivable functions with:

- the sign of $\tilde{g}^{(0)}$ and of $G_i(x)$ coincide in all points of M ,
- for $1 \leq j \leq m'$, the sign of $\tilde{g}^{(j)}$ and of $\frac{d\tilde{g}^{(j-1)}}{dt}$ coincide in all points of M , and
- function $\tilde{g}^{(m')}$ has a constant sign over M .

Proof. For $i = 0$, this is easy as $\text{Val}_0 = 0$, so we are left with a polynom P for which the statement holds.

Now consider some I and assume it is the case for all $i < I$.

We unfold Val_I to express it wrt Val_{I-1} , multiply by a degree 2-polynomials PP that is always positive (similar to how it is done above) then derive $N + 3$ times (N being the maximum degree among $P, P_0^\alpha, \dots, P_K^\alpha, P_0^\beta, \dots, P_K^\beta$) to get rid of integrals and fall back on a

term of our general form with the action value Val_{I-1} . We are left with a function $GG_{i-1}(x)$ of the generic form above, for which we can apply the induction hypothesis to obtain a $mm \in \mathbb{N}$ and a sequence $(\tilde{g}^{(j)})_{j=0}^{mm}$ of derivable functions satisfying the statement.

It suffices then to fix $m' = mm + N + 3$, $\tilde{g}^{(0)}(x) = G_i(x)PP(x)$, which has the same sign as $G_i(x)$ over all x , set $\tilde{g}^{(j)}(x)$ the j -derivative of $\tilde{g}^{(0)}(x)$ for all $j \leq N + 2$, and $\tilde{g}^{(j+N+2)} = \tilde{g}^{(j)}$ for all $j \leq mm$. ◀

Applying this result, it suffices to choose $m = m' + 2$ and $\tilde{f}^{(j)} = \tilde{g}^{(j-2)}$ for all $2 \leq j \leq m' + 2$. This solves the case where all distributions are uniform distributions. We now consider the easiest case:

B.1.2 Case: All distributions on delays are exponential distributions

This is the easiest case, as exponential distributions integrates and derives nicely. Since all distributions on delays are exponential, we can write Val_{i+1} as

$$\begin{aligned} \text{Val}_{i+1}(\ell, x, \iota) &= \mathcal{R}(\iota) + \gamma \int_{x \leq t < x_1^t} e^{-\lambda t} \text{Val}_i(\ell, t) dt \\ &\quad + \gamma \sum_{1 \leq j \leq J_\iota} \left(\int_{x_j^t \leq t < x_{j+1}^t} e^{-\lambda_j t} \text{Val}_i(\ell, t) dt \right) \end{aligned}$$

We can thus write Val_i under the general form:

$$\text{Val}_i(\ell, x, \iota) = C_0 + C_1 e^{-\lambda_\iota x}$$

for C_0, C_1 constants and $\iota \in \{\alpha, \beta\}$, which is shown by direct induction over i .

Further, $f(x) = \text{Val}_i(\ell, x, \alpha) - \text{Val}_i(\ell, x, \beta)$ is of the form $C_0 + C_1 e^{-\lambda_\alpha x} + C_2 e^{-\lambda_\beta x}$. We can show that functions of this form satisfy the statement by the following process:

1. derive the term to get rid of C_0 ;
2. divide the term by one of the $e^{-\lambda x}$ (an always non-negative factor, so this does not change the sign of the term) ;
3. derive the term and return to step **2** till there is no more exponential term.

This procedure leaves us with a polynomial term R , which we derive sufficiently to obtain a constant. It is a n -th derivative of our original term up to multiplication by non-negative factors along the way. This solves the case where all distributions are exponential distributions.

We are thus left with the general case combining the ideas of the previous cases:

B.1.3 General case with both exponential and uniforms distributions

To study the sign of $f_i = \text{Val}_i(\ell, x, \alpha) - \text{Val}_i(\ell, x, \beta)$, we multiply it by one or two polynomial depending on the number of uniform distributions encountered.

For example, if the transition for α follows a uniform distribution and the one for β follows an exponential distribution, we study:

$$\begin{aligned} (x_{J_{\alpha+1}}^\alpha - x)f(x) &= C(x_{J_{\alpha+1}}^\alpha - x) + C' \int_{x \leq t < x_m} \text{Val}_i(\ell_\alpha, t) dt + \\ &\quad C''(x_{J_{\alpha+1}}^\alpha - x) \int_{x \leq t < x_m} e^{-\lambda_\beta t} \text{Val}_i(\ell_\beta, t) dt \end{aligned}$$

By definition, $x < x_{J_{\alpha+1}}^\alpha$, hence the term $(x_{J_{\alpha+1}}^\alpha - x)$ is always positive. The second derivative of $(x_{J_{\alpha+1}}^\alpha - x)f(x)$ has the following form:

$$C_1 + C_2 \text{Val}_i(\ell_\alpha, x) + P_1 \text{Val}_i^{(1)}(\ell_\alpha, x) + P_2 e^{-\lambda x} \text{Val}_i^{(1)}(\ell_\beta, x)$$

for C_1, C_2 constants and P_1, P_2 polynomials.

Inductively, we will obtain terms of the general form:

$$g_i(x) = \sum_{k \leq K} P_k(x) e^{-\lambda_k x} \text{Val}_i^{(k)}(\ell_k, x)$$

for $(\ell_k)_{k \in K}$ control states, $(\lambda_k)_{k \in K}$ constants (can be 0) and $(P_k)_{k \in K}$ polynomials.

We now prove by induction on $i \in \mathbb{N}$ similarly as above that for all g_i of this form, the statement holds.

Proof. The statement is true for $i = 0$ as $v_0 = 0$.

We suppose it is true for $I \in \mathbb{N}$. We express Val_{I+1} in terms of Val_I and get rid of the integrals as described above by multiplying by the right always positive polynomial and deriving once or twice. Then we perform the following:

1. divide the term by one of the $e^{-\lambda_k t}$ associated with an integral (an always non-negative factor, so this does not change the sign of the term) ;
2. derive the term enough to get rid of this integral (degree of the associated polynomial +1 times) and return to step 1 until there is no more integrals.

The loop removes an integral at each step, and does not create more exponentials (it only replaces some already existing exponentials with different ones).

This solves the case where distributions are a mix of both distributions. ◀

B.1.4 Resets on the transitions

In this proof, we assumed there were no resets for the studied transitions. If there is a reset for the transition taken when playing α in a mode, then $v_i(\ell, x, \alpha)$ is a constant of x on considered mode. It is easily seen that it eases most calculus and all arguments can be made all the same.

B.2 Proof of Lemma 21

Before proceeding to the proof of Lemma 21, we first provide an important technical lemma bounding the derivative IVal'_i of IVal_i .

► **Lemma 23.** *$\text{IVal}_i(\ell, x)$ is derivable, with $\text{IVal}'_i(\ell, x) \leq \sum_{j \leq i} \gamma^j \text{MaxReward} \leq \text{MaxValue}$, for all $x \in (a, b)$.*

Proof. We show the result by induction over $i \geq 0$. For $i = 0$, it is trivial as $\text{Val}_i(\ell, t) = 0$ for all t .

Assume the result is true for some $i \in \mathbb{N}$. We show that it is true for $i + 1$, by considering a small deviation εc for $c = b - x$. We let $\varphi(t) = (1 + \varepsilon)t - \varepsilon b$. We have $\varphi(b) = b$, and $\varphi(x + \varepsilon c) = x$ as $c = b - x$. We will proceed to a change of variable t for $\varphi(t)$. We have

$$g_{i+1}(\ell, x) = \frac{1}{c} \int_x^b \text{Val}_{i+1}(\ell, t) dt = \frac{1}{c} \int_{x+\varepsilon c}^b (1 + \varepsilon) \text{Val}_{i+1}(\ell, (1 + \varepsilon)t - \varepsilon b) dt$$

Thus, we have:

$$g_{i+1}(\ell, x + \varepsilon c) - g_{i+1}(\ell, x) = \int_{x+\varepsilon c}^b \frac{1}{c(1 - \varepsilon)} \text{Val}_{i+1}(\ell, t) - \frac{1 + \varepsilon}{c} \text{Val}_{i+1}(\ell, (1 + \varepsilon)t' - \varepsilon b) dt$$

Notice that $\frac{1}{c(1-\varepsilon)} = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{c}$, so the different with $\frac{1+\varepsilon}{c}$ is $O(\varepsilon^2)$. Let us focus on the term proportional to $\frac{1+\varepsilon}{c}$, as the other one will be negligible. Now, we use:

$$\text{Val}_{i+1}(\ell, x) = \mathcal{R}(\sigma_{i+1}(\ell, x)) + \gamma g_i(x).$$

For the rewards, $\int_{x+\varepsilon c}^b \mathcal{R}(\sigma_{i+1}(\ell, t)) dt - \int_{x+\varepsilon c}^b \mathcal{R}(\sigma_{i+1}(\ell, (1+\varepsilon)t' - \varepsilon b)) dt$, the reward is piecewise constant (constant on each mode of σ_{i+1}), and the value of the integral is the sum over all modes of size of the mode times the reward constant. All the rewards in $\mathcal{R}(\sigma_{i+1}(\ell, (1+\varepsilon)t' - \varepsilon b))$ can be matched in $\mathcal{R}(\sigma_{i+1}(\ell, t))$, besides the initial rewards from x to $x + \varepsilon c$ which are partly compensated by other rewards > 0 . This accounts for a difference of at most $\varepsilon c \text{MaxReward}$.

Concerning the second term, we use the inductive hypothesis providing us with the derivation factor. It tells us that for α tending towards 0, we have

$$\left| \frac{g_i(x + \alpha) - g_i(x)}{\alpha} \right| \rightarrow \sum_{j \leq i} \gamma^j \text{MaxReward}.$$

Now, we have a bound on the difference $\alpha = |(1+\varepsilon)t' - \varepsilon b - t| \leq c\varepsilon$. We thus have $|\frac{1+\varepsilon}{c} \gamma \int_{x+\varepsilon c}^b g_i(\ell, t) - g_i(\ell, (1+\varepsilon)t - \varepsilon b) dt| \leq \frac{1+\varepsilon}{c} \gamma \int_{x+\varepsilon c}^b c\varepsilon \sum_{j \leq i} \gamma^j \text{MaxReward} dt = (1+\varepsilon)(1-\varepsilon)\gamma c\varepsilon \sum_{j \leq i} \gamma^j \text{MaxReward}$.

Thus, $g'_{i+1}(x) = \sum_{j \leq i+1} \gamma^j \text{MaxReward}$ as when $\varepsilon \rightarrow 0$, the derivation limit is:

$$\frac{g_{i+1}(x + \varepsilon c) - g_{i+1}(x)}{\varepsilon c} \rightarrow \sum_{j \leq i+1} \gamma^j \text{MaxReward}$$

The proof for unbounded $I(\ell) = (a, \infty)$ is simpler as exponentials are behaving in a simpler way. ◀

We are now ready to prove Lemma 21, that we restate here:

► **Lemma 20.** *Let f and m be as defined in Lemma 19. Then for all $\varepsilon_0 > 0$, one can compute $b_0 \leq a_1 \leq b_1 \leq \dots \leq a_m$ such that for all $j \leq m$, f does not change sign on $[b_j, a_{j+1}]$, and $|f(t)| \leq \varepsilon_1 = 5\varepsilon_0 \text{MaxValue}$ over all $t \in [a_j, b_j]$ and $|b_j - a_j| \leq \varepsilon_0$.*

Lemma 19 provides a m and functions $(\tilde{f}^{(j)})_{j=0}^m$ such that $\tilde{f}^{(m)}$ does not change sign. Also, $\tilde{f}^{(0)}(x)$ and $f(x)$ have the same sign for all x .

We show by induction on $j \leq m$ that we can obtain $b_0^j \leq a_1^j \leq b_1^j \leq \dots \leq a_{m-j+1}^j$ such that for all $j \leq m$, $\tilde{f}^{(j)}$ does not change sign on $[b_i^j, a_{i+1}^j]$, and $|\tilde{f}^{(j)}(t)| \leq 5\varepsilon_0 \text{MaxValue}$ over all $t \in (a_i^j, b_i^j)$ and $|b_i^j - a_i^j| \leq \varepsilon_0$.

Proof. We let $(b, a) = I(\ell)$. We proceed inductively on j from $j = m$ to $j = 0$ for $\tilde{f}^{(j)}$, to define $b_0^j \leq a_1^j \leq b_1^j \leq \dots \leq a_{m-j+1}^j$. First, $\tilde{f}^{(m)}$ has no change of sign by hypothesis, and we set $b_0^m = b, a_1^m = a$. Say $\tilde{f}^{(m)}$ is positive.

Hence the derivative of $\tilde{f}^{(m-1)}$ is also positive, and $\tilde{f}^{(m-1)}$ is increasing, thus it has at most one change of sign. We evaluate $\tilde{f}^{(m-1)}$ at the boundaries b, a to see whether the signs of $\tilde{f}^{(m-1)}(b)$ and of $\tilde{f}^{(m-1)}(a)$ are opposite. If not, then $\tilde{f}^{(m-1)}$ does not change its sign and we can set $b_0^{m-1} = b, a_1^{m-1} = a$. Otherwise, there is a change of sign and we use Ridder's method [Rid79] (alternatively, ITP Interpolate, Truncate and Project [OT21]) to iteratively shrink the interval (a_1, b_1) until we obtain $|b_1 - a_1| \leq \varepsilon_0$ where the root of $\tilde{f}^{(m-1)}$ lies. We then set $b_0^{m-1} = b, a_1^{m-1} = a_1, b_1^{m-1} = b_1, a_2^{m-1} = a$.

To evaluate $\tilde{f}^{(m-1)}(x)$ for different values $x \in [b, a]$, we resort to numerical integration (Newton-Cotes and Generalized Gauss-Laguerre quadrature) in order to evaluate both the variables in the expression of $\tilde{f}^{(m-1)}$ as mentioned in the proof of Lemma 19, as well as the function $\tilde{f}^{(m-1)}(x)$ itself, with a precision ensuring that the sign of $\tilde{f}^{(m-1)}(x)$ is determined. Indeed, for any values $x_1 < x_2$, as $\tilde{f}^{(m-1)}(x_1) < \tilde{f}^{(m-1)}(x_2)$, at least one of them cannot be zero. By iteratively increasing the precision, we will eventually obtain bounds $\alpha_1 < \tilde{f}^{(m-1)}(x_2) < \beta_1$ and $\alpha_2 < \tilde{f}^{(m-1)}(x_2) < \beta_2$ such that α_k, β_k have the same sign, for at least one of $k = 1, 2$. Thus, one can evaluate the sign of $\tilde{f}^{(m-1)}(x)$ for all except at most 1 value x , which is not a restriction to obtain the desired a_i, b_i with precision ε_0 .

We repeat inductively the process for $\tilde{f}^{(m-2)}$ over (b_0^{m-1}, a_1^{m-1}) and over (b_1^{m-1}, a_2^{m-1}) . We know that $\tilde{f}^{(m-2)}$ has at most 2 roots, so if we find one root in (b_0^{m-1}, a_1^{m-1}) between (a_1, b_1) and one root in (b_1^{m-1}, a_2^{m-1}) between (a_2, b_2) using Ridder's method with precision ε_0 , then we can set $b_0^{m-2} = b \leq a_1^{m-2} = a_1 < b_1^{m-2} = b_1 < a_2^{m-2} = a_2 < b_2^{m-2} = b_2 \leq a_3^{m-2} = a$ and we are done.

Otherwise, there are several cases:

- If $\tilde{f}^{(m-2)}(b) < 0$ or $\tilde{f}^{(m-2)}(a_1) > 0$, there is no root of $\tilde{f}^{(m-2)}$ in (b_0^{m-1}, a_1^{m-1}) as it is decreasing,
- Similarly, if $\tilde{f}^{(m-2)}(a) < 0$ or $\tilde{f}^{(m-2)}(b_1) > 0$, there is no root in (b_1^{m-1}, a_2^{m-1})
- Thus, if both of the above hold, then roots can only be in (a_1^{m-1}, b_1^{m-1}) (there is 0 or 2 roots here - we may not be able to determine which case holds, only the parity, which is fine because the interval is already small), and we set $b_0^{m-2} = b \leq a_1^{m-2} = a_1^{m-1} \leq b_1^{m-2} = b_1^{m-1} = a_2^{m-2} = b_2^{m-2} \leq a_3^{m-2} = a$
- Else, exactly one of the two above cases is true, in which case there is one root that can be found using Ridder's method in the other interval, plus possibly a root in (a_1^{m-1}, b_1^{m-1}) which is already small enough (considering the sign of $\tilde{f}^{(m-2)}(a_1), \tilde{f}^{(m-2)}(b_1)$ suffices to know which case holds). In both case we can set the appropriate $b = b_0^{m-1} \leq a_1^{m-1} \leq b_1^{m-1} \leq a_2^{m-1} \leq b_2^{m-1} \leq a_3^{m-2} = a$

We inductively reach $f = \tilde{f}^{(0)}$ itself. We have $f(x) = \mathcal{R}(\alpha) - \mathcal{R}(\beta) + \text{IVal}_i(\ell, x, \alpha) - \text{IVal}_i(\ell, x, \beta)$. We apply Lemma 23 on both $\text{IVal}_i(\ell, x, \alpha)$ and $\text{IVal}_i(\ell, x, \beta)$ to obtain that the derivative f' of f is bounded by 2MaxValue as $\mathcal{R}(\alpha) - \mathcal{R}(\beta)$ is a constant.

For all i , if $|f(a_i)| < 3 \text{MaxValue} \varepsilon_0$, then as $|b_i - a_i| < \varepsilon_0$, we have that $f(t) < 3 \text{MaxValue} \varepsilon_0 + 2 \text{MaxValue} \varepsilon_0 = 5 \text{MaxValue} \varepsilon_0$ for all $t \in (a_i, b_i)$. On another hand, if $|f(a_i)| > 2 \text{MaxValue} \varepsilon_0$, then we know that $f(t) > 0$ for all $t \in (a_i, b_i)$, and we can remove (a_i, b_i) by integrating $(b_{i-1}, a_i), (a_i, b_i), (b_i, a_{i+1})$ together in (b_{i-1}, a_i) , setting $a_i := a_{i+1}$.

By evaluating $f(a_i)$ with precision $\text{MaxValue} \varepsilon_0$, we can decide which one of these cases holds for sure (and may be both, in which case we remove (a_i, b_i)). In both cases, (a_i, b_i) follows the requirements. \blacktriangleleft