

Foerster 2018

$f: \mathbb{R} \rightarrow \mathbb{R}$ est 2π -périodique et égale à $\pi^2 t - t^3 \sin$ sur $[-\pi, \pi]$

$$I_{k,n} = \int_0^\pi t^k \sin(nt) dt$$

$$\textcircled{1} I_{1,n} = \int_0^\pi t \sin(nt) dt$$

$$\text{I.I.I} \left| \begin{array}{l} u = t \quad u' = 1 \\ v' = \sin(nt) \quad v = -\frac{\cos(nt)}{n} \end{array} \right.$$

$$I_{1,n} = \left[-\frac{t \cos(nt)}{n} \right]_0^\pi + \int_0^\pi \frac{\cos(nt)}{n} dt$$

$$= -\frac{\pi \cos(n\pi)}{n} + 0 + \left[\frac{\sin(nt)}{n^2} \right]_0^\pi = \frac{-\pi(-1)^n}{n} = \frac{(-1)^{n+1} \pi}{n}$$

$$\textcircled{2} I_{k,n} = \int_0^\pi t^k \sin(nt) dt$$

$$\text{I.I.I} \left| \begin{array}{l} u = t^k \quad u' = k t^{k-1} \\ v' = \sin(nt) \quad v = -\frac{\cos(nt)}{n} \end{array} \right.$$

$$I_{k,n} = \left[-t^k \frac{\cos(nt)}{n} \right]_0^\pi + \int_0^\pi k t^{k-1} \frac{\cos(nt)}{n} dt$$

$$\text{I.I.I} \left| \begin{array}{l} u = k t^{k-1} \quad u' = k(k-1) t^{k-2} \\ v' = \frac{\cos(nt)}{n} \quad v = \frac{\sin(nt)}{n^2} \end{array} \right.$$

$$I_{k,n} = -\pi \frac{k \cos(n\pi)}{n} + \left[\frac{k}{n^2} t^{k-1} \frac{\sin(nt)}{n} \right]_0^\pi - \int_0^\pi \frac{k(k-1)}{n^2} t^{k-2} \sin(nt) dt = \frac{k(-1)^{n+1} \pi}{n} - \frac{k(k-1)}{n^2} I_{k-2,n}$$

$$\textcircled{3} I_{3,n} = \pi \frac{3}{n} (-1)^{n+1} - \frac{3 \times 2}{n^2} I_{1,n} = \pi \frac{3}{n} (-1)^{n+1} - \frac{6}{n^2} (-1)^{n+1} \frac{\pi}{n}$$

$$I_{3,n} = \pi \frac{3}{n} (-1)^{n+1} + \frac{6\pi (-1)^n}{n^3}$$

$$\textcircled{4} \int_0^\pi f(t) \sin(nt) dt = \int_0^\pi (\pi^2 t - t^3) \sin(nt) dt$$

$$= \pi^2 \int_0^\pi t \sin(nt) dt - \int_0^\pi t^3 \sin(nt) dt$$

$$= \pi^2 I_{1,n} - I_{3,n}$$

$$= \pi^2 \frac{(-1)^{n+1} \pi}{n} - \left(\pi \frac{3}{n} (-1)^{n+1} + \frac{6\pi (-1)^n}{n^3} \right)$$

$$\int_0^\pi f(t) \sin(nt) dt = \frac{6\pi (-1)^{n+1}}{n^3}$$

$$f(-t) = \pi^2(-t) - (-t)^3 = -\pi^2 t + t^3 = -(\pi^2 t - t^3) = -f(t)$$

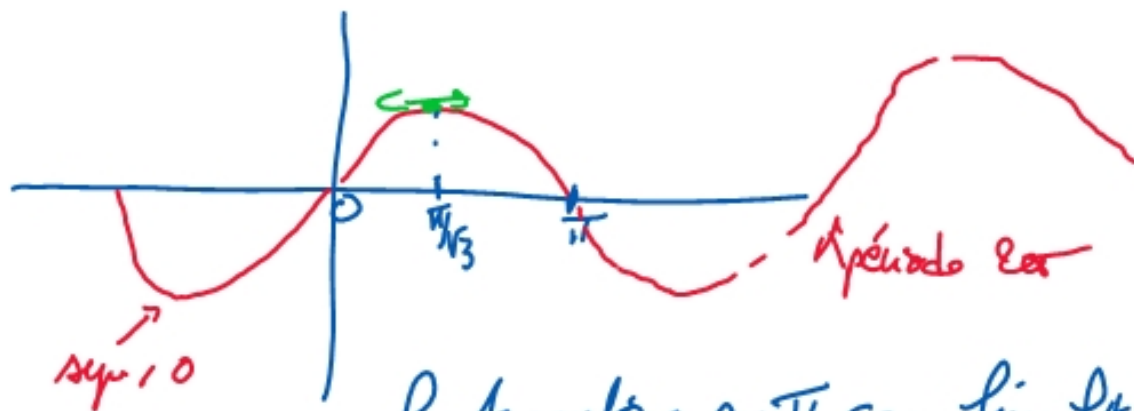
f est impaire (sym / 0)

$$f'(t) = \pi^2 - 3t^2 \quad f'(t) = 0 \Leftrightarrow t^2 = \frac{\pi^2}{3} \Leftrightarrow t = \pm \frac{\pi}{\sqrt{3}}$$

t	0	$\frac{\pi}{\sqrt{3}}$	π
$f'(t)$	+	0	-
$f(t)$	0	$\frac{2\pi^3}{3\sqrt{3}}$	0

+ sym / 0

$$f\left(\frac{\pi}{\sqrt{3}}\right) = \frac{\pi^3}{\sqrt{3}} - \frac{\pi^3}{3\sqrt{3}} = \frac{2\pi^3}{3\sqrt{3}}$$



(3)

f est continue en π car $\lim_{t \rightarrow \pi^-} f(t) = 0 = \lim_{t \rightarrow \pi^+} f(t)$.

4a) $a_n = 0$ car f est impaire

4b)
$$b_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt$$
$$= \frac{1}{\pi} \times 2 \int_0^{\pi} f(t) \sin(nt) dt \quad \text{car } f \text{ est impaire}$$

$$b_n = \frac{2}{\pi} \times \frac{6\pi (-1)^{n+1}}{n^3} = \frac{12(-1)^{n+1}}{n^3}$$

⑤ T de Dirichlet: f est 2π -périodique et de classe C^1 par morceaux, donc $S_f(x)$ converge $\frac{f(x^+) + f(x^-)}{2}$, $\forall x$.

Ici f est continue sur \mathbb{R} , donc $S_f(x) = f(x)$, $\forall x$

⑥
$$S_f\left(\frac{\pi}{2}\right) = \sum_{n=1}^{+\infty} \frac{12(-1)^{n+1}}{n^3} \sin\left(n \frac{\pi}{2}\right) = \sum_{n \text{ pair}} + \sum_{n \text{ impair}}$$

$$\begin{aligned}
 f\left(\frac{\pi}{2}\right) &= \sum_{p=1}^{+\infty} \frac{12(-1)^{2p+1}}{(2p)^3} \sin\left(\frac{2p\pi}{2}\right) + \sum_{p=0}^{+\infty} \frac{12(-1)^{2p+2}}{(2p+1)^3} \sin\left(\frac{(2p+1)\pi}{2}\right) \\
 &= \sum_{p=0}^{+\infty} \frac{12 \sin\left(p\pi + \frac{\pi}{2}\right)}{(2p+1)^3} = \sum_{p=0}^{+\infty} \frac{12(-1)^p \sin\left(\frac{\pi}{2}\right)}{(2p+1)^3}
 \end{aligned}$$

$$f\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = \pi^2 \cdot \frac{\pi}{2} - \left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{2} - \frac{\pi^3}{8} = \frac{3\pi^3}{8}$$

$$12 \sum_{p=0}^{+\infty} \frac{(-1)^p}{(2p+1)^3} = \frac{3\pi^3}{8} \Rightarrow \sum_{p=0}^{+\infty} \frac{(-1)^p}{(2p+1)^3} = \frac{3\pi^3}{8 \times 12} = \frac{\pi^3}{32}$$

$$(7) \int_0^{\pi} f'(t) dt = \int_0^{\pi} (\pi^2 t - t^3) dt = \int_0^{\pi} (\pi^4 t^2 - 2\pi^2 t^4 + t^6) dt$$

$$= \left[\frac{\pi^4 t^3}{3} - 2\pi^2 \frac{t^5}{5} + \frac{t^7}{7} \right]_0^{\pi} = \frac{\pi^7}{3} - \frac{2\pi^7}{5} + \frac{\pi^7}{7} = \frac{35 - 42 + 15}{105} \pi^7 = \frac{8\pi^7}{105}$$

$$(8) \text{Parseval: } \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt = \frac{1}{2} \sum_{n \in \mathbb{N}} |b_n|^2 \quad \text{Fin}$$