

① Somme partielle d'une suite géométrique

$$1+x+\dots+x^n = \frac{x^{n+1}-1}{x-1} \text{ qd } x \neq 1 \text{ et nH qd } x=1$$

② $\cos(x-y) = \cos x \cos y + \sin x \sin y$

③ dérivée de $\arctan(x)$ $\arctan' x = \frac{1}{1+x^2}$

④ DL de $\sin x$ à l'ordre 4 $\sin(x) = x - \frac{x^3}{6} + o(x^4)$

⑤ Matrice de passage de la base \mathcal{A} à la base \mathcal{B}

$$P_{\mathcal{A} \rightarrow \mathcal{B}} = \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \\ \text{dans} & \text{dans} & \text{dans} \\ \mathcal{A} & \mathcal{A} & \mathcal{A} \end{pmatrix} \text{ où } \mathcal{B} = (\vec{u}, \vec{v}, \vec{w})$$

$$\begin{aligned}
 f(t) &= 16 \cos^2\left(\frac{3t}{2}\right) \sin^2\left(\frac{t}{2}\right) + 16 \sin^2\left(\frac{3t}{2}\right) \sin^2\left(\frac{t}{2}\right) \\
 &= 16 \sin^2\left(\frac{t}{2}\right) \left[\cos^2\left(\frac{3t}{2}\right) + \sin^2\left(\frac{3t}{2}\right) \right] = \underline{16 \sin^2\left(\frac{t}{2}\right)}
 \end{aligned}$$

$$P(x) = -x^3 + 3x + 2 \quad P(1) = 4 \quad P(-1) = 1 - 3 + 2 = 0$$

-1 est racine de P , donc on peut factoriser par $X+1$

$$\begin{array}{r|l}
 -x^3 + 3x + 2 & x+1 \\
 \hline
 -(-x^3 - x^2) & \\
 \hline
 x^2 + 3x + 2 & \\
 -(x^2 + x) & \\
 \hline
 2x + 2 & \\
 -(2x + 2) & \\
 \hline
 0 &
 \end{array}$$

$\Delta = 9$

$$\begin{aligned}
 P(x) &= -(x+1)(x-2)(x+1) \\
 P(x) &= -(x+1)^2(x-2)
 \end{aligned}$$

$$\text{2 racis. } \frac{-1 \pm 3}{-2} = \begin{matrix} +2 \\ -1 \end{matrix}$$

$$P(x) = (x+1)^2(2-x)$$

$$P(x) = -x^3 + 2x^2 + ax - 2a$$

$$P(1) = -1 + 2 + a - 2a = a - a = 0$$

$$P(2) = -8 + 8 + 2a - 2a = 0$$

$$(-\sqrt{a})^2 = a$$

$$\sqrt{a^2} = |a| = \begin{cases} a & \text{si } a \geq 0 \\ -a & \text{si } a < 0 \end{cases}$$

$$\begin{array}{r} -x^3 + 2x^2 + ax - 2a \\ -(-x^3 + 2x^2) \\ \hline ax - 2a \\ - (ax - 2a) \\ \hline 0 \end{array}$$

$$P(x) = (x-2)(a-x^2) \\ = (x-2)(\sqrt{a}^2 - x^2)$$

$$P(x) = (x-2)(\sqrt{a}-x)(\sqrt{a}+x)$$

$$Q(t) = -3 \sin t + 6 \cos^2 t \sin t = \sin t \cdot 3[-1 + 2\cos^2 t]$$

$$= 3 \sin t [2\cos^2 t - 1] = 3 \sin t \cos(2t)$$

$$\left(\begin{array}{l} \sin t = 0 \Leftrightarrow t = k\pi \text{ ou } k \in \mathbb{Z} \\ \cos(2t) = 0 \Leftrightarrow 2t = \frac{\pi}{2} + k\pi \\ t = \frac{\pi}{4} + k\frac{\pi}{2} \end{array} \right)$$

$$\begin{aligned} P(X) = X^4 - 2X^2 + 1 &= Y^2 - 2Y + 1 \quad \text{avec } X^2 = Y \\ &= (Y-1)^2 = (X^2-1)^2 \\ &= ((X-1)(X+1))^2 = (X-1)^2(X+1)^2 \end{aligned}$$

$$u_0 = -1 \text{ et } u_1 = \frac{1}{2}$$

$$u_{n+2} = u_{n+1} - \frac{1}{4} u_n$$

$$u_2 = u_1 + \frac{1}{4} u_0 = \frac{1}{2} + \frac{1}{4} = \frac{2+1}{4} = \frac{3}{4}$$

$$\frac{u_2}{u_1} = \frac{\frac{3}{4}}{\frac{1}{2}} = \frac{3}{4} \times 2 = \frac{3}{2} \quad \text{et} \quad \frac{u_1}{u_0} = \frac{\frac{1}{2}}{-1} = -\frac{1}{2}$$

$\frac{u_2}{u_1} \neq \frac{u_1}{u_0}$, donc la suite est pas géométrique

$u_2 - u_1 \neq u_1 - u_0$, donc la suite n'est pas arithmétique

$$\text{2) } v_n = u_{n+1} - \frac{1}{2} u_n \quad \underline{v_0 = 1}$$

$$v_{n+1} = u_{n+2} - \frac{1}{2} u_{n+1} = u_{n+1} - \frac{1}{4} u_n - \frac{1}{2} u_{n+1} = \frac{1}{2} u_{n+1} - \frac{1}{4} u_n$$

$$v_{n+1} = \frac{1}{2} \left(u_{n+1} - \frac{1}{2} u_n \right) = \frac{1}{2} v_n. \quad (v_n)_{n \in \mathbb{N}} \text{ est géométrique de raison } \frac{1}{2}$$

$$\Rightarrow U_n = \left(\frac{1}{2}\right)^n \cdot U_0 = \underline{\frac{1}{2^n}}$$

$$\omega_n = \frac{U_{n+1}}{U_n} \quad \omega_0 = -1$$

$U_n = \frac{1}{2^n} \neq 0$, donc ω_n est bien définie

$$\omega_{n+1} = \frac{U_{n+1}}{U_{n+1}} = \frac{U_n + \frac{1}{2}U_n}{\frac{1}{2}U_n} = \frac{U_n + \frac{1}{2}U_n}{\frac{1}{2}U_n} = \frac{U_n + \frac{1}{2}U_n}{\frac{1}{2}U_n} = \frac{2U_n + U_n}{U_n} = \frac{3U_n}{U_n} = 3$$

$$\omega_{n+1} = \frac{U_n + \frac{1}{2}U_n}{\frac{1}{2}U_n} = 2 + \omega_n$$

donc (ω_n) est arithmétique de raison 2

$$\omega_n = 2n + \omega_0 = \underline{2n - 1}$$

$$(4) U_n = U_0 \cdot \omega_n = \frac{1}{2^n} \cdot (2n - 1)$$

M par retourner que $S_n = 2 - \frac{2n+3}{2^n}$

$$S_n = \sum_0^n u_k = u_0 + u_1 + \dots + u_n$$

Hercule: Je prends un $n \geq 1$ Je suppose que $S_n = 2 - \frac{2n+3}{2^n}$

$$S_{n+1} = \boxed{u_0 + u_1 + \dots + u_n} + u_{n+1}$$

$$= 2 - \frac{2n+3}{2^n} + \frac{2n+1}{2^{n+1}} = 2 - \left(\frac{4n+6}{2^{n+1}} - \frac{2n+1}{2^{n+1}} \right)$$

$$u_n = \frac{2n-1}{2^n} \Rightarrow u_{n+1} = \frac{2n+1}{2^{n+1}}$$

$$S_{n+1} = 2 - \frac{2n+5}{2^{n+1}}$$

FIN

$$u_{n+2} - u_{n+1} + \frac{1}{4}u_n = 0$$

$$X^2 - X + \frac{1}{4} \quad \text{a double root } \frac{1}{2}$$

$$\text{Sols } u_n = (A + B) \frac{1}{2^n}$$

III Fractions trigonométriques

ex 1: $I = \int \frac{1}{1+3\cos^2 x} dx$

le chgt $t = \tan\left(\frac{x}{2}\right)$ conduit à l'intégrale d'une fraction rationnelle

$$\sin(x) = \frac{2 \tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$$

$$\sin(x) = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$$

$$\cos(x) = \frac{1-t^2}{1+t^2}$$

où

$$t = \tan\left(\frac{x}{2}\right)$$

$$t = \tan\left(\frac{x}{2}\right)$$

$$\frac{dt}{dx} = \left(1 + \tan^2\left(\frac{x}{2}\right)\right) \times \frac{1}{2} = (1+t^2) \frac{1}{2}$$

$$\Rightarrow dx = \frac{2}{1+t^2} dt$$

$$I = \int \frac{1}{1+3\cos^2 x} dx = \int \frac{1}{1+3\left(\frac{1-t^2}{1+t^2}\right)^2} \frac{2}{1+t^2} dt$$

$$I = \int \frac{2}{1+t^2 + 3\frac{(1-t^2)^2}{1+t^2}} dt \quad \times \frac{1+t^2}{1+t^2}$$

$$= \int \frac{2(1+t^2)}{(1+t^2)^2 + 3(1-t^2)^2} dt = 2 \int \frac{t^2+1}{t^4+2t^2+1+3t^4-6t^2+3} dt$$

$$= 2 \int \frac{t^2+1}{4t^4-4t^2+4} dt = \frac{1}{2} \int \frac{t^2+1}{t^4-t^2+1} dt$$

Il faut réduire ça en des simples

On a envie de faire autrement

Méthode des invariants

aque $\cos(-x) = \cos(x)$

\cos est "invariante" qd on remplace x par $-x$

$$\sin(\pi - x) = \sin x$$

\sin est "invariante" qd on remplace x par $\pi - x$



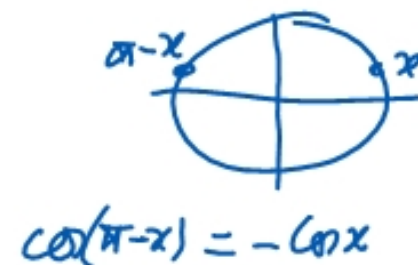
\tan est invariante qd on remplace x par $\pi + x$

Si on veut calculer $\int F(\cos x, \sin x) dx$ où F est une fraction, alors
on remplace x par $-x$ dans $F(\cos x, \sin x) dx$)
_____ $\pi - x$)
_____ $\pi + x$)
si c'est invariant)
chgt $t = \cos x$
chgt $t = \sin x$
chgt $t = \tan x$

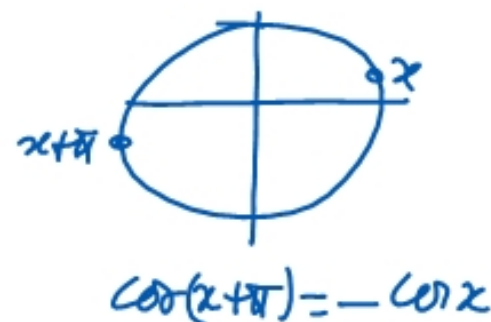
Ex 1: $I = \int \frac{1}{1+3\cos^2 x} dx$ On cherche un u tel que

$$\frac{1}{1+3\cos^2(-x)} d(-x) = \frac{-dx}{1+3\cos^2 x} \quad \text{NON}$$

$$\frac{1}{1+3\cos^2(\pi-x)} d(\pi-x) = \frac{1}{1+3\cos^2(x)} (-dx) \quad \text{NON}$$



$$\frac{1}{1+3\cos^2(x+\pi)} d(x+\pi) = \frac{1}{1+3\cos^2(x)} dx \quad \text{YES}$$



chgt $t = \tan(x)$ $\frac{dt}{dx} = 1 + \tan^2(x) = 1+t^2 \Rightarrow dx = \frac{dt}{1+t^2}$

$$\tan^2(x) = \frac{1}{\cos^2(x)} = 1 + \tan^2(x) \Rightarrow \cos^2(x) = \frac{1}{1+t^2}$$

$$\Rightarrow I = \int \frac{1}{1+3\frac{1}{1+t^2}} \frac{dt}{1+t^2}$$

$$I = \int \frac{1}{1+t^2+3} dt = \int \frac{1}{t^2+4} dt \quad (\text{c'est bcp + simple.})$$

$$= \frac{1}{4} \int \frac{1}{\frac{t^2}{4}+1} dt = \frac{1}{4} \int \frac{1}{\left(\frac{t}{2}\right)^2+1} dt \quad \text{dlt } T = \frac{t}{2} \quad \frac{dT}{dt} = \frac{1}{2}$$

$\Rightarrow dt = 2dT$

$$= \frac{1}{4} \int \frac{1}{T^2+1} 2dT = \frac{1}{2} \operatorname{arctan}(T) + cste$$

$$= \frac{1}{2} \operatorname{arctan}\left(\frac{t}{2}\right) + cste$$

$$= \frac{1}{2} \operatorname{arctan}\left(\frac{\tan(\alpha)}{2}\right) + cste$$

ex2: $I = \int \frac{|\sin(x)|}{1+\cos^2(x)} dx$

On remplace x par $-x$: $\frac{\sin(-x)}{1+\cos^2(-x)} d(-x) = \frac{-\sin(x)}{1+\cos^2(x)} (-dx) = \frac{\sin(x)}{1+\cos^2(x)} dx$

choix $t = \cos x$ $\frac{dt}{dx} = -\sin x \Rightarrow dx = \frac{dt}{-\sin x}$

Da

$I = \int \frac{\cancel{\sin(x)}}{1+t^2} \frac{dt}{\cancel{-\sin(x)}} = -\arctan(t) + cste = -\arctan(\cos x) + cste$

ex3: $I = \int \frac{1}{1+\tan(x)} dx$

Invariants: $x \rightarrow -x$ NON
 $x \rightarrow \pi+x$ HAi

choix $x = \tan x$ $\frac{dx}{dx} = 1+\tan^2(x) = 1+x^2 \Rightarrow dx = \frac{dx}{1+x^2}$

$I = \int \frac{1}{(1+x)(1+x^2)} dx$

$I = \int \frac{1}{(1+x)(1+x^2)} dx$

$$\frac{1}{(1+x)(1+x^2)} = \frac{a}{1+x} + \frac{bx+c}{x^2+1} + E(x)$$

Esito cal...

$$a = \frac{1}{1+x^2} \Big|_{x=-1} = \frac{1}{2} \quad \cdot \text{On moltiplica per } x^2+1: \frac{1}{1+x} = \frac{a(x^2+1)}{1+x} + bx+c$$

$$\text{valore in } i: \frac{1}{1+i} = bi+c$$

$$\frac{1-i}{(1+i)(1-i)} = bi+c \Rightarrow \frac{1-i}{2} = bi+c \Rightarrow b = -\frac{1}{2} \text{ e } c = \frac{1}{2}$$

$$I = \int \frac{1/2}{1+x} dx + \int \frac{-1/2 x}{x^2+1} dx + \int \frac{1/2}{x^2+1} dx$$

$$= \frac{1}{2} \ln|1+x| - \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \arctan x + cte$$

$\arctan(\tan x) = x$

$$\underline{I = \frac{1}{2} \ln|1+\tan(x)| - \frac{1}{4} \ln(\tan^2(x)+1) + \frac{1}{2} x + cte}$$

$$\text{ex 4: } I = \int \frac{1}{1+\sin(x)} dx$$

$$x \rightarrow -x : \frac{1}{1-\sin x} \cdot -dx = \frac{dx}{-1+\sin x} \quad \text{NO}$$

$$x \rightarrow \pi - x \quad \frac{1}{1+\sin(x)} (-dx) \quad \text{NO}$$

$$x \rightarrow \pi + x \quad \frac{1}{1-\sin x} dx \quad \text{NO}$$

Smiff.

$$\text{chgt } t = \tan\left(\frac{x}{2}\right)$$

$$\frac{dt}{dx} = \left(1 + \tan^2\left(\frac{x}{2}\right)\right) \frac{1}{2} \Rightarrow \boxed{dx = \frac{2}{1+t^2} dt}$$

$$\sin(x) = \frac{2t}{1+t^2}$$

$$I = \int \frac{1}{1 + \frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2} dt = 2 \int \frac{1}{1+t^2+2t} dt = 2 \int \frac{1}{t^2+2t+1} dt$$

$$I = 2 \int \frac{1}{(t+1)^2} dt = 2 \left(-\frac{1}{t+1} \right) + C = \underline{\underline{-\frac{2}{1+\tan\left(\frac{x}{2}\right)} + C}}$$