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$$D = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$\begin{matrix} l_1+l_2+l_3 \\ = \\ = \end{matrix} \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = \begin{vmatrix} a+b+c & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

$l_2-l_1 \quad l_3-l_1$

$$= (a+b+c)(-a-b-c)(-a-b-c) = (a+b+c)^3$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

$$A^n = ?$$

$$K = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} = \text{Mat}_B(\phi)$$

$$\vec{u} = (\sqrt{a}, 0, -\sqrt{b})$$

$$\text{M. que } f(\vec{u}) = -\sqrt{ab} \vec{u}$$

$$f(\vec{u}) = K \cdot \begin{pmatrix} \sqrt{a} \\ 0 \\ -\sqrt{b} \end{pmatrix} = \begin{pmatrix} -a\sqrt{b} \\ 0 \\ b\sqrt{a} \end{pmatrix}$$

$$-\sqrt{ab} \vec{u} = -\sqrt{ab} \begin{pmatrix} \sqrt{a} \\ 0 \\ -\sqrt{b} \end{pmatrix} = \begin{pmatrix} -a\sqrt{b} \\ 0 \\ b\sqrt{a} \end{pmatrix} \quad \text{OK!}$$

$$A^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 8 & 0 & 5 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 8 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 0 & 7 \\ 0 & 1 & 0 \\ 28 & 0 & 13 \end{pmatrix}$$

2) Trouver $\vec{v} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\text{K} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ avec $y=1$

$$K \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} az \\ 0 \\ bx \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} az=0 \\ bx=0 \end{cases} \Rightarrow \begin{cases} z=0 \\ x=0 \end{cases} \quad \underline{\underline{\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}}$$

$f(\vec{v}) = \vec{0}$, car $\vec{v} \in \underline{\underline{\text{Ker}(f)}}$

③ Trouver $\vec{\omega} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ t que $f(\vec{\omega}) = \sqrt{ab} \vec{\omega}$ avec $\chi_{\vec{\omega}} = \sqrt{a}$

$$K \begin{pmatrix} \sqrt{a} \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{a} \\ y \\ z \end{pmatrix} = \begin{pmatrix} az \\ 0 \\ \sqrt{ab} \end{pmatrix} = \sqrt{ab} \vec{\omega} = \sqrt{ab} \begin{pmatrix} \sqrt{a} \\ y \\ z \end{pmatrix} = \begin{pmatrix} a\sqrt{b} \\ \sqrt{ab}y \\ \sqrt{ab}z \end{pmatrix}$$

$$\Rightarrow \begin{cases} az = a\sqrt{b} \\ 0 = \sqrt{ab}y \\ \sqrt{ab} = \sqrt{ab}z \end{cases} \Rightarrow \begin{cases} z = \sqrt{b} \\ y = 0 \\ z = \sqrt{b} \end{cases} \quad \underline{\vec{\omega} = (\sqrt{a}; 0; \sqrt{b})}$$

④ $B' = (\vec{u}, \vec{v}, \vec{\omega})$

$$\begin{vmatrix} \sqrt{a} & 0 & \sqrt{b} \\ 0 & 1 & 0 \\ -\sqrt{b} & 0 & \sqrt{b} \end{vmatrix} \begin{matrix} \text{col 1} \\ \text{col 2} \\ \text{col 3} \end{matrix} \begin{matrix} \vec{u} \\ \vec{v} \\ \vec{\omega} \end{matrix} = \pm \begin{vmatrix} \sqrt{a} & \sqrt{a} \\ -\sqrt{b} & \sqrt{b} \end{vmatrix} = 2\sqrt{ab} \neq 0$$

donc B' est une base de \mathbb{R}^3

$$P_{B \rightarrow B'} = \begin{pmatrix} \text{coord de } \vec{u} & & \\ \text{de } \vec{v} & & \\ \text{de } \vec{\omega} & & \end{pmatrix} = \begin{pmatrix} \sqrt{a} & 0 & \sqrt{a} \\ 0 & 1 & 0 \\ -\sqrt{b} & 0 & \sqrt{b} \end{pmatrix} = \underline{P}$$

$$\text{Mat}_{\mathcal{B}}(\mathcal{P}) = \begin{pmatrix} \text{coord} \\ \text{de} \\ \mathcal{P}(\vec{a}), \mathcal{P}(\vec{b}), \mathcal{P}(\vec{c}) \\ \text{ds } \mathcal{B}' \end{pmatrix}$$

$$\mathcal{P}(\vec{a}) = \sqrt{ab} \vec{u} = \sqrt{ab} \begin{pmatrix} \sqrt{a} \\ 0 \\ -\sqrt{b} \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} \sqrt{ab} \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}'}$$

$$\mathcal{P}(\vec{b}) = \vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}'}$$

$$\text{Mat}_{\mathcal{B}'}(\mathcal{P}) = \begin{pmatrix} \sqrt{ab} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{ab} \end{pmatrix}$$

$$\mathcal{P}(\vec{c}) = \sqrt{ab} \vec{w} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{ab} \end{pmatrix}_{\mathcal{B}'}$$

⑦ Si $\mathcal{P}(\vec{a}) = \lambda \vec{a}$, alors $(\alpha \mathcal{P} + \beta \text{Id}_{\mathbb{R}^3})(\vec{a}) = (\alpha \lambda + \beta) \vec{a}$

$$(\alpha \mathcal{P} + \beta \text{Id}_{\mathbb{R}^3})(\vec{a}) = \alpha \mathcal{P}(\vec{a}) + \beta \text{Id}_{\mathbb{R}^3}(\vec{a}) = \alpha \lambda \vec{a} + \beta \vec{a} = (\alpha \lambda + \beta) \vec{a}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \text{ et } I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = K + I \quad \text{avec } a=1 \quad b=4$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad b=4 \quad a=1 \quad \alpha=\beta=1$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ devient dans } \mathcal{B}' \quad \begin{pmatrix} -\sqrt{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{4} \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ ————— } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = K + I \text{ ————— } \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix} \begin{matrix} \vec{u} \\ \vec{v} \\ \vec{u} + \vec{w} \end{matrix}$$

$$Q^{-1} = ?$$

dans Q , on a $\vec{u}, \vec{v}, \vec{w}$ en fait $\vec{i}, \vec{j}, \vec{k}$
 dans Q^{-1} , on a $\vec{i}, \vec{j}, \vec{k}$ ——— $\vec{u}, \vec{v}, \vec{w}$

col 3: $2\vec{u} = \vec{u} + \vec{w} \Rightarrow \vec{u} = \frac{1}{2}\vec{u} + \frac{1}{2}\vec{w} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$

col 2: $\vec{j} = \vec{v} \Rightarrow \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

col 1: $\vec{i} - 2\vec{k} = \vec{u} \Rightarrow 2\vec{k} = \vec{i} - \vec{u} = \frac{1}{2}\vec{u} + \frac{1}{2}\vec{w} - \vec{u} = -\frac{1}{2}\vec{u} + \frac{1}{2}\vec{w}$

$$\Rightarrow \vec{k} = -\frac{1}{4}\vec{u} + \frac{1}{4}\vec{w} = \begin{pmatrix} -\frac{1}{4} \\ 0 \\ \frac{1}{4} \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix}$$

$$\textcircled{10} \Delta = Q^{-1} A Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \Delta^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^n \end{pmatrix}$$

$$A^n = Q \Delta^n Q^{-1}$$

Montrer que $A^n = Q \Delta^n Q^{-1}$ par récurrence -

pour $n=1$: $A = Q \Delta Q^{-1}$ car $\Delta = Q^{-1} A Q$

formule des carrés

$$Q \Delta = Q Q^{-1} A Q = I_3 A Q = A Q$$

$$Q \Delta Q^{-1} = A Q Q^{-1} = A I_3 = A$$

$$Q \Delta Q^{-1} = A \quad \text{OK}$$

Propriété. Je prends un $n > 1$. Je suppose que $A^n = Q \Delta^n Q^{-1}$

$$A^{n+1} = A^n \cdot A = Q \Delta^n Q^{-1} Q \Delta Q^{-1} = Q \Delta^n I \Delta Q^{-1} = Q \Delta^{n+1} Q^{-1} = Q \Delta^{n+1} Q^{-1}$$

$$A^{n+1} = Q \Delta^{n+1} Q^{-1}$$

FIN

ex2: Diagonaliser la matrice $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$

$$P(x) = \begin{vmatrix} 1-x & 0 & 1 \\ 0 & 1-x & 0 \\ 4 & 0 & 1-x \end{vmatrix} \stackrel{\text{det}}{=} -0 + (1-x) \begin{vmatrix} 1-x & 1 \\ 4 & 1-x \end{vmatrix} - 0$$

$$= (1-x) [0-x^2-4] = (1-x) [(1-x-2)(1-x+2)]$$

$$= (1-x) (-1-x)(3-x)$$

3 valeurs propres 1, -1, et 3 (A est fsi, on aura $A' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$)

Pour la valeur propre -1: $E_{-1} = \left\{ \vec{u} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid f(\vec{u}) = -1 \vec{u} \right\}$

$$f(\vec{u}) = -\vec{u} \Leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} \Leftrightarrow \begin{cases} x+z = -x \\ y = -y \\ 4x+z = -z \end{cases} \Leftrightarrow \begin{cases} 2x+z = 0 \\ 2y = 0 \\ 4x+2z = 0 \end{cases} \Leftrightarrow \begin{cases} 2x+z = 0 \\ y = 0 \end{cases}$$

un vecteur de E_{-1} est $\vec{u} = (1; 0; -2)$

pour la valeur propre 1: $E_1 = \left\{ \vec{v} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 / f(\vec{v}) = 1 \vec{v} \right\}$

$$f(\vec{v}) = 1 \vec{v} \Leftrightarrow \begin{cases} x+z = x \\ y = y \\ 4x+z = z \end{cases} \Leftrightarrow \begin{cases} z=0 \\ 0=0 \\ x=0 \end{cases}$$

un vecteur de E_1 est

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

pour la valeur propre 3: $E_3 = \left\{ \vec{w} \begin{pmatrix} x \\ y \\ z \end{pmatrix} / f(\vec{w}) = 3 \vec{w} \right\}$

$$f(\vec{w}) = 3 \vec{w} \Leftrightarrow A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix} \Leftrightarrow \begin{cases} x+z = 3x \\ y = 3y \\ 4x+z = 3z \end{cases} \Leftrightarrow \begin{cases} -2x+z=0 \\ -2y=0 \Leftrightarrow y=0 \\ 4x-2z=0 \end{cases}$$

un vecteur de E_3 est $\vec{w} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\mathcal{B} = (\vec{i}, \vec{j}, \vec{k})$ est la base canonique de \mathbb{K}^3 et $\mathcal{B}' = (\vec{u}, \vec{v}, \vec{w})$ est une autre base

$$\text{Car } \det_{\mathcal{B}}(\vec{u}, \vec{v}, \vec{w}) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 2 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} = 4$$

$$\text{Mat}_{\mathcal{B}}(f) = \begin{pmatrix} f(\vec{u}) \\ f(\vec{v}) \\ f(\vec{w}) \end{pmatrix} = A'$$

$$\text{or } f(\vec{u}) = -\vec{u} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}'}$$

$$f(\vec{v}) = \vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{\mathcal{B}'}, \text{ et } f(\vec{w}) = 3\vec{w} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}_{\mathcal{B}'}$$

$$A' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$