

ORAL 2017 Exo 1

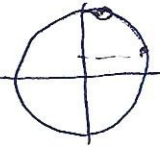
① a) Avec  $X_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$  et  $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$   $X_{n+1} = AX_n$   
 C'est une suite géométrique, donc  $X_n = A^n X_0$

b)  $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ . On trouve  $N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  et  $N^3 = 0$ , donc  $N^p = 0, \forall p \geq 3$ .

c)  $A^n = (3I + N)^n = \sum_{k=0}^n \binom{n}{k} N^k (3I)^{n-k} = \binom{n}{0} (3I)^n + \binom{n}{1} N (3I)^{n-1} + \binom{n}{2} N^2 (3I)^{n-2}$   
 $= 3^n I + n 3^{n-1} N + \frac{n(n-1)}{2} 3^{n-2} N^2 = \begin{pmatrix} 3^n & n 3^{n-1} & \frac{n(n-1)}{2} 3^{n-2} \\ 0 & 3^n & n 3^{n-1} \\ 0 & 0 & 3^n \end{pmatrix}$

d) Donc  $X_n = A^n(X_0) = A^n \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} 3^n a_0 + n 3^{n-1} b_0 + \frac{n(n-1)}{2} 3^{n-2} c_0 \\ 3^n b_0 + n 3^{n-1} c_0 \\ 3^n c_0 \end{pmatrix}$   $\leftarrow a_n$   
 $\leftarrow b_n$   
 $\leftarrow c_n$

②  $\cos^2 x - \sin^2 x = \sin(3x) = \sin(x+2x) = \sin x \cos(2x) + \cos x \sin(2x)$   
 $= \sin x (1 - 2\sin^2 x) + \cos x (2\sin x \cos x) = \sin x - 2\sin^3 x + 2\sin x \cos^2 x$   
 $1 - 2\sin^2 x = 3\sin x - 3\sin^3 x \Leftrightarrow 3\sin^3 x - 2\sin^2 x - 3\sin x + 1 = 0$

On pose  $X = \sin x$   $3X^3 - 2X^2 - 3X + 1 = 0$  BOF   
 $\cos(2x) = \cos^2 x - \sin^2 x$ . On obtient  $\cos(2x) = \sin(3x)$   
 $= \cos(\frac{\pi}{2} - 3x)$   
 $2x = \pm (\frac{\pi}{2} - 3x) + 2k\pi$   $\xrightarrow{1^{\text{er}} \text{ cas}}$   $2x = \frac{\pi}{2} - 3x + 2k\pi \Rightarrow x = \frac{\pi}{10} + \frac{2k\pi}{5}$  où  $k \in \mathbb{Z}$   
 $\xrightarrow{2^{\text{e}} \text{ cas}}$   $2x = -\frac{\pi}{2} + 3x + 2k\pi \Rightarrow x = \frac{\pi}{2} + 2k\pi$  où  $k \in \mathbb{Z}$

③  $I = \int_0^1 x (\operatorname{arctan} x)^2 dx$ .  $\int \int \left| \begin{array}{l} u = \operatorname{arctan} x \\ v' = x \end{array} \right. \quad \begin{array}{l} u' = \frac{\operatorname{arctan} x}{1+x^2} \\ v = \frac{x^2}{2} \end{array}$   
 $I = \left[ \frac{x^2}{2} \operatorname{arctan}^2 x \right]_0^1 - \int_0^1 \frac{x^2}{1+x^2} \operatorname{arctan} x dx$   $\int \int \left| \begin{array}{l} u = \operatorname{arctan} x \\ v' = \frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2} \end{array} \right. \quad \begin{array}{l} u' = \frac{1}{1+x^2} \\ v = x - \operatorname{arctan} x \end{array}$   
 Rappel:  $\operatorname{arctan} 1 = \frac{\pi}{4}$   
 $I = \frac{1}{2} \frac{\pi^2}{16} - [x \operatorname{arctan} x - \operatorname{arctan}^2 x]_0^1 + \int_0^1 \frac{x}{1+x^2} dx - \int_0^1 \frac{\operatorname{arctan} x}{1+x^2} dx$  J  
 J se calcule aussi avec une  $\int \int$ :  $\left| \begin{array}{l} u = \operatorname{arctan} x \\ v' = \frac{1}{1+x^2} \end{array} \right. \quad \begin{array}{l} u' = \frac{1}{1+x^2} \\ v = \operatorname{arctan} x \end{array}$   
 $J = [\operatorname{arctan} x]_0^1 = J \Rightarrow 2J = \frac{\pi^2}{16}$   $J = \frac{\pi^2}{32}$ . De plus,  $\int_0^1 \frac{x}{1+x^2} dx = \left[ \frac{1}{2} \ln(1+x^2) \right]_0^1 = \frac{\ln 2}{2}$   
 $I = \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{\pi^2}{16} + \frac{\ln 2}{2} - \frac{\pi^2}{32}$

$I = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{\ln 2}{2}$