

ATS 901 (WACHE Thimoké)

$$(1) S(x) = \sum_{n=0}^{+\infty} (n^2 + 1) x^{2n} \quad a_n = (n^2 + 1) x^{2n}$$

$$\left| \frac{a_{n+1}}{a_n} \right| \underset{n \rightarrow +\infty}{\sim} \frac{(n+1)^2 x^{2n+2}}{n^2 x^{2n}} \underset{n \rightarrow +\infty}{\sim} x^2$$

Si $x^2 < 1$, $S(x)$ CV (D'Alambert)
Si $x^2 > 1$, $S(x)$ D.V

Donc $R=1$ On travaille sur $]-1, 1[$.

$$\sum_0^{+\infty} x^{2n} = \frac{1}{1-x^2}, \text{ donc } \boxed{\sum_0^{+\infty} x^{2n} = \frac{1}{1-x^2}} \text{ On dérive: } \sum_0^{+\infty} 2n x^{2n-1} = \frac{2x}{(1-x^2)^2}$$

On multiplie par x : $\sum_0^{+\infty} 2n x^{2n} = \frac{2x^2}{(1-x^2)^2}$

On dérive: $\sum_0^{+\infty} 2n^2 x^{2n-1} = \frac{2x(1-x^2)^2 + 2(1-x^2)2x}{(1-x^2)^4} = \frac{2x(1-x^2) + 4x}{(1-x^2)^3}$
 $= \frac{6x - 2x^3}{(1-x^2)^3}$

On multiplie par x :
et on dérive par 2

$$\boxed{\sum_0^{+\infty} 2n^2 x^{2n} = \frac{3x^2 - x^4}{(1-x^2)^3}}$$

$$S(x) = \sum_0^{+\infty} n^2 x^{2n} + \sum_0^{+\infty} x^{2n} = \frac{3x^2 - x^4}{(1-x^2)^3} + \frac{1}{1-x^2}$$

$$= \frac{3x^2 - x^4 + 1 - 2x^2 + x^4}{(1-x^2)^3} = \frac{x^2 + 1}{(1-x^2)^3}$$

② $A = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$ $B = (\vec{u}, \vec{v})$ est la base canonique de \mathbb{R}^2

$$A = \text{Mat}_B(f)$$

$$\vec{u}(y) \in \ker(f) \Leftrightarrow f(\vec{u}) = \vec{0} \Leftrightarrow \begin{cases} -2x + 2y = 0 \\ 2x - 2y = 0 \end{cases} \Leftrightarrow x - y = 0$$

$$\boxed{\ker(f) = \text{Vect} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)}$$

ensemble de départ

$$\dim(\text{Im}(f)) + \dim(\ker(f)) = \dim(\mathbb{R}^2) \quad \text{donc} \quad \dim(\text{Im}(f)) = 1$$

$f\left(\begin{pmatrix} -2 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ donc $\begin{pmatrix} -2 \\ 2 \end{pmatrix} \in \text{Im}(f)$. On peut aussi choisir $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\boxed{\text{Im}(f) = \text{Vect} \left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)}$$

Si $\vec{u} \in \ker(f) \cap \text{Im}(f)$, alors $\vec{u} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ et $\vec{u} = b \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{cases} a = -b \\ a = b \end{cases} \Rightarrow a = b = 0 \quad \text{Donc} \quad \boxed{\ker(f) \cap \text{Im}(f) = \{ \vec{0} \}}$$

$$\text{De plus, } \boxed{\dim(\ker(f)) + \dim(\text{Im}(f)) = \dim(\mathbb{R}^2)} \quad \text{donc} \quad \boxed{\mathbb{R}^2 = \ker(f) \oplus \text{Im}(f)}$$

$$\vec{u} \in \ker(f) \Leftrightarrow f(f(\vec{u})) = \vec{0} \Leftrightarrow A^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix}$$

$$A^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 8x - 8y = 0 \\ -8x + 8y = 0 \end{cases} \Leftrightarrow x - y = 0$$

$$\boxed{\ker(f \circ f) = \ker(f)}$$