

ATS 2021 (GUICHARD Julien)

① $\lim_{x \rightarrow 0^+} \frac{\sin(x \ln x)}{x} = ?$

$x \ln x \xrightarrow{x \rightarrow 0^+} 0$ (T des croissances comparées)

donc $\sin(x \ln x) \underset{0^+}{\sim} x \ln x$ (car $\sin(x) \underset{x \rightarrow 0}{\sim} x$)

donc $\frac{\sin(x \ln x)}{x} \underset{0^+}{\sim} \frac{x \ln x}{x} = \ln x \xrightarrow{x \rightarrow 0^+} -\infty$

② $(z+1)^n = (z-1)^n \Leftrightarrow \left(\frac{z+1}{z-1}\right)^n = 1$ ($z=1$ n'est pas solution. Sinon $z^n = 0$)

On pose $Z = \frac{z+1}{z-1}$ $Z^n = 1$, donc $Z_k = e^{i \frac{2k\pi}{n}}$ où $k \in \mathbb{Z}$
 $k=0, 1, \dots, n-1$ suffisent

$\frac{Z_k^n}{Z_{k-1}^n} = e^{i \frac{2k\pi}{n}} \Leftrightarrow Z_k^n = Z_{k-1}^n e^{i \frac{2k\pi}{n}} = e^{i \frac{2(k-1)\pi}{n}} e^{i \frac{2k\pi}{n}} = e^{i \frac{2k\pi}{n}} + 1 = Z_k (e^{i \frac{2k\pi}{n}} - 1)$

$\Leftrightarrow Z_k = \frac{e^{i \frac{2k\pi}{n}} + 1}{e^{i \frac{2k\pi}{n}} - 1}$ sauf qd $k=0$ (sinon, on divise par 0 !)

$Z_k = \frac{e^{i \frac{2k\pi}{n}} + 1}{e^{i \frac{2k\pi}{n}} - 1} = \frac{e^{i \frac{k\pi}{n}} (e^{i \frac{k\pi}{n}} + e^{-i \frac{k\pi}{n}})}{e^{i \frac{k\pi}{n}} (e^{i \frac{k\pi}{n}} - e^{-i \frac{k\pi}{n}})} = \frac{2 \cos\left(\frac{k\pi}{n}\right)}{2i \sin\left(\frac{k\pi}{n}\right)} = -i \frac{\cos\left(\frac{k\pi}{n}\right)}{\sin\left(\frac{k\pi}{n}\right)}$

Bonus: $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = ?$

$\forall x \neq 0, -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

$\Rightarrow x \sin\left(\frac{1}{x}\right)$ est compris entre $-x$ et x

$\lim_{x \rightarrow 0} \pm x = 0$, donc $\boxed{\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0}$ (est le T des gendarmes)