

Chapitre

- ① Somme partielles géométrique
 - ② DL de $\ln(1-x)$ à l'ordre n $\ln(1-x)$
 - ③ Formule des binômes $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^n \binom{n}{n-k} a^{n-k} b^k$
 - ④ $\cos(2x) = ?$ de 2 façons \neq
 - ⑤ $\binom{15}{2} = ?$ $\binom{15}{2} = \frac{15!}{2!(15-2)!} = \frac{\cancel{2} \cdot \cancel{13} \cdot 14 \cdot 15}{9 \cdot \cancel{2} \cdot \dots \cdot \cancel{13}}$
 $\cos 2x = 2\cos^2 x - 1 \Rightarrow \overset{13!}{\cancel{12}} = 7 \cdot 15 = 105$
 $= 1 - 2\sin^2 x$ $\ln(1-x) = -x - \frac{x^2}{2} - \dots - \frac{x^n}{n} + o(x^n)$
- $1 + x + \dots + x^n = \begin{cases} \frac{x^{n+1} - 1}{x - 1} & \text{qd } x \neq 1 \\ n+1 & \text{qd } x = 1 \end{cases}$

$$I = \int e^x \cos(e^x) dx \quad \text{chgt } t = e^x \quad \frac{dt}{dx} = e^x \Rightarrow dx = \frac{dt}{e^x} = \frac{dt}{t}$$

$$I = \int \cancel{t} \cos t \frac{dt}{\cancel{t}} = \sin t + c_1 = \sin(e^x) + c_1$$

$$J = \int \frac{\sin(\ln x)}{x} dx \quad \text{chgt } t = \ln x \quad \frac{dt}{dx} = \frac{1}{x} \Rightarrow dx = x dt$$

$$J = \int \frac{\sin(t)}{\cancel{x}} \cancel{x} dt = -\cos t = -\cos(\ln x) + c_1$$

$$\int \cos(37x) dx = \frac{\sin(37x)}{37} + c_1$$

$$I = \int \cos x \sin^3 x dx \quad \text{chgt } t = \sin x \quad \frac{dt}{dx} = \cos x \rightarrow dx = \frac{dt}{\cos x}$$

$$I = \int \cancel{\cos x} t^3 \frac{dt}{\cancel{\cos x}} = \frac{t^4}{4} + c_1 = \frac{\sin^4 x}{4} + c_1$$

$$I = \int \cos x \sin^3 x \, dx$$

$$I = \cos x \sin^3 x \int 1 \, dx = \cos x \sin^3 x$$

$$I = \int t e^{t^2} \, dt \quad \text{chgt } x = t^2 \quad \frac{dx}{dt} = 2t \Rightarrow dt = \frac{dx}{2t}$$

$$I = \int \cancel{t} e^x \frac{dx}{\cancel{2t}} = \frac{e^x}{2} + cste = \frac{e^{t^2}}{2} + cste$$

$$k = \int \frac{1}{x^3} \, dx = \int x^{-3} \, dx = \frac{x^{-2}}{-2} + cste = -\frac{1}{2x^2} + cste$$

$$I = \int \frac{x}{(x^2+1)^3} \, dx \quad \text{chgt } t = x^2+1 \quad \frac{dt}{dx} = 2x \Rightarrow dx = \frac{dt}{2x}$$

$$I = \int \frac{\cancel{x}}{t^3} \frac{dt}{\cancel{2x}} = \frac{1}{2} \int t^{-3} \, dt = \frac{1}{2} \left(+ \frac{t^{-2}}{-2} \right) + cste = -\frac{1}{4t^2} + cste$$

$$(9) \quad I = \int \frac{\cos t}{\sin^3 t} dt \quad x = \sin t \quad \frac{dx}{dt} = \cos t \Rightarrow dt = \frac{dx}{\cos t}$$

$$I = \int \frac{\cancel{\cos t}}{x^3} \frac{dx}{\cancel{\cos t}} = \int x^{-3} dx = -\frac{1}{2x^2} + cste = -\frac{1}{2\sin^2 t} + cste$$

$$(10) \quad J = \int \frac{\cos t}{\sin^2 t + 1} dt \quad \text{chgt } x = \sin t \quad dt = \frac{dx}{\cos t}$$

$$J = \int \frac{\cancel{\cos t}}{x^2 + 1} \frac{dx}{\cancel{\cos t}} = \arctan x + cste = \arctan(\sin t) + cste.$$

$$\int \frac{1}{x^2 + 1} dx = \arctan x \quad (\arctan)' = \frac{1}{x^2 + 1}$$

(26) Obtention des DLs

La formule de la somme partielle géométrique donne $\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \mathcal{O}(x^{n+1})$

En remplaçant x par $-x$, on a $\frac{1}{1+x} = 1 - x + x^2 + \dots + (-1)^n x^n + \mathcal{O}(x^{n+1})$.

En intégrant, on a $\ln(1+x)$ et $\ln(1-x)$

Taylor en 0 pour $f(x) = e^x$ $f'(x) = e^x$ $f''(x) = e^x$ $f^{(n)}(x) = e^x$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \mathcal{O}(x^{n+1})$$

Avec $f(x) = e^x$: $e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \mathcal{O}(x^{n+1})$ car $f^{(n)}(0) = 1, \forall n$

$$\operatorname{ch}x = \frac{e^x + e^{-x}}{2}$$

$$\operatorname{sh}x = \frac{e^x - e^{-x}}{2}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} + \dots + \frac{(-1)^n x^n}{n!} + \mathcal{O}(x^{n+1})$$

$$e^x = 1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} + \cancel{\frac{x^5}{5!}} + \dots$$

$$e^{-x} = 1 - \cancel{x} + \frac{x^2}{2!} - \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} - \cancel{\frac{x^5}{5!}} + \dots$$

$$e^x + e^{-x} = 2 + 2 \cdot \frac{x^2}{2!} + 2 \cdot \frac{x^4}{4!} + \dots$$

$$\frac{e^x + e^{-x}}{2} = \cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + \mathcal{O}(x^{2n+1})$$

$$e^x = \cancel{x} + x + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \cancel{\frac{x^4}{4!}} + \frac{x^5}{5!} + \dots$$

$$e^{-x} = \cancel{x} - x + \cancel{\frac{x^2}{2!}} - \frac{x^3}{3!} + \cancel{\frac{x^4}{4!}} - \frac{x^5}{5!} + \dots$$

$$e^x - e^{-x} = 2x + 2 \cdot \frac{x^3}{3!} + 2 \cdot \frac{x^5}{5!} + \dots$$

$$\frac{e^x - e^{-x}}{2} = \sinh(x) = x + \frac{x^3}{3!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + \mathcal{O}(x^{2n+2})$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \dots$$

$$e^{ix} = 1 + \cancel{ix} - \frac{x^2}{2!} - \cancel{i \frac{x^3}{3!}} + \frac{x^4}{4!} + \cancel{i \frac{x^5}{5!}} - \frac{x^6}{6!} - \cancel{i \frac{x^7}{7!}} + \frac{x^8}{8!} + \dots$$

$$e^{-ix} = 1 - \cancel{ix} - \frac{x^2}{2!} + \cancel{i \frac{x^3}{3!}} + \frac{x^4}{4!} - \cancel{i \frac{x^5}{5!}} - \frac{x^6}{6!} + \cancel{i \frac{x^7}{7!}} + \frac{x^8}{8!} + \dots$$

$$e^{ix} + e^{-ix} = 2 - \frac{2 \cdot x^2}{2!} + \frac{2 \cdot x^4}{4!} - \frac{2 \cdot x^6}{6!} + \frac{2 \cdot x^8}{8!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \mathcal{O}(x^{2n}) \quad (\text{ou } \mathcal{O}(x^{2n+1}))$$

$$e^{ix} - e^{-ix} = 2ix - \frac{2ix^3}{3!} + \frac{2ix^5}{5!} - \frac{2ix^7}{7!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \mathcal{O}(x^{2n+2}) \quad \text{ou } \mathcal{O}(x^{2n+3})$$

Assume $f(x) = (1+x)^\alpha$ $f'(x) = \alpha(1+x)^{\alpha-1}$ $f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$

$$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$$

$$f(0) = 1^\alpha = 1 \quad f'(0) = \alpha \cdot 1 = \alpha \quad f''(0) = \alpha(\alpha-1) \quad f'''(0) = \alpha(\alpha-1)(\alpha-2)$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!} + \dots$$

⇓

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)x^2}{2!} + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)x^3}{3!} + \dots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2)$$

avec $f(x) = \ln(1-x)$ $f'(x) = -\frac{1}{1-x}$ $f'(0) = -\left(-\frac{1}{(1-0)^2} \cdot -1\right)$

$\left(\frac{1}{x}\right)' = (x^{-1})' = -\frac{1}{x^2}$ $\left(\frac{1}{u}\right)' = -\frac{1}{u^2} \cdot u'$ $f''(x) = -\frac{1}{(1-x)^2}$

$f'''(x) = ?$ $\left(\frac{1}{x^2}\right)' = (x^{-2})' = -2x^{-3} \Rightarrow \left(\frac{1}{u^2}\right)' = -2u^{-3} \cdot u'$

$\left(\frac{1}{(1-x)^2}\right)' = -\frac{2}{(1-x)^3} \cdot (-1)$ $f'''(x) = \left(-\frac{1}{(1-x)^2}\right)' = -\left[\frac{2}{(1-x)^3} \cdot (-1)\right]$

$f'''(x) = -\frac{2}{(1-x)^3}$

$f''''(x) = -2 \left(\frac{1}{(1-x)^3}\right)' = -2 \cdot (-3) \frac{1}{(1-x)^4} \cdot (-1)$

$f''''(x) = -\frac{2 \cdot 3}{(1-x)^4}$

$f''''(x) = -\frac{2 \cdot 3 \cdot 4}{(1-x)^5}$

$f(0) = \ln(1) = 0$ $f'(0) = -1$ $f''(0) = -1$ $f'''(0) = -2$ $f''''(0) = -2 \cdot 3$

$f^{(5)}(0) = -2 \cdot 3 \cdot 4$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \dots$$

$$\ln(1-x) = 0 - 1x - \frac{1}{2!}x^2 - \frac{2}{3!}x^3 - \frac{2 \cdot 3}{4!}x^4 - \frac{2 \cdot 3 \cdot 4}{5!}x^5 + \dots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \dots$$

$$\left(\text{ici } f^{(n)}(x) = -\frac{(n-1)!}{(1-x)^n} \right)$$

III les théorèmes de comparaison des croissances

ex1: $\lim_{x \rightarrow +\infty} \frac{e^x}{x^{37}} = \lim_{x \rightarrow +\infty} e^x = +\infty$ (c'était une F.V. : $\frac{\infty}{\infty}$)

ex2: $\lim_{x \rightarrow +\infty} \frac{x^{37}}{\ln(x)} = \lim_{x \rightarrow +\infty} x^{37} = +\infty$ (c'est une F.V. $\frac{\infty}{\infty}$)

$$\boxed{\text{I}} \quad \lim_{x \rightarrow +\infty} \frac{\ln x}{x^n} = 0, \quad \forall n > 0$$

$$\Leftrightarrow \lim_{x \rightarrow +\infty} \frac{x^n}{\ln x} = +\infty \quad \forall n > 0$$

II On étudie $f(x) = \frac{\ln x}{x^{n/2}}$

$$f'(x) = \frac{\frac{1}{x} x^{n/2} - \frac{n}{2} x^{n/2-1} \ln x}{(x^{n/2})^2}$$

$$f'(x) = \frac{x^{n/2-1} - \frac{n}{2} x^{n/2-1} \ln x}{x^n}$$

$$= \frac{x^{n/2-1} [1 - \frac{n}{2} \ln x]}{x^n}$$

$f'(x)$ est du signe de $g(x) = 1 - \frac{n}{2} \ln x$

$$g(x) = 0 \Leftrightarrow \frac{n}{2} \ln x = 1 \Leftrightarrow \ln x = \frac{2}{n} = x = e^{\frac{2}{n}}$$

x	0	$e^{2/n}$	$+\infty$
$f'(x)$		+	-
$f(x)$		$\nearrow \frac{2}{ne}$	\searrow

$$f(e^{2/n}) = \frac{\ln(e^{2/n})}{(e^{2/n})^{n/2}} = \frac{2/n}{e} = \frac{2}{ne}$$

et $\forall x > 1, \ln(x) > 0$, donc $f(x) > 0$

$$\forall x \geq e^{2/n} : 0 < f(x) < \frac{2}{ne} \quad \dots \boxed{\text{I}}$$