

$$\textcircled{1} u_0 = 1 \quad u_{n+1} = \frac{u_n - 6}{u_n - 4} \quad v_n = \frac{u_n - 3}{u_n - 2}$$

$$v_{n+1} = \frac{u_{n+1} - 3}{u_{n+1} - 2} = \frac{\frac{u_n - 6}{u_n - 4} - 3}{\frac{u_n - 6}{u_n - 4} - 2} \quad \begin{array}{l} \times (u_n - 4) \\ \times (u_n - 4) \end{array}$$

$$v_{n+1} = \frac{u_n - 6 - 3(u_n - 4)}{u_n - 6 - 2(u_n - 4)} = \frac{-2u_n - 6 + 12}{-u_n - 6 + 8} = \frac{6 - 2u_n}{2 - u_n} = 2 \frac{(3 - u_n)}{(2 - u_n)}$$

$$= 2 \frac{u_n - 3}{u_n - 2} = 2 v_n$$

$v_{n+1} = 2 v_n$, $\forall n \in \mathbb{N}$ Donc $(v_n)_{n \in \mathbb{N}}$ est une suite géométrique

$$v_n = 2^n \cdot v_0 = 2^n \cdot \frac{u_0 - 3}{u_0 - 2} = 2^n \cdot \frac{-2}{-1} = 2^{n+1} \quad \text{do raison } 2$$

$$v_n = 2^{n+1}$$

$$v_n = 2^{n+1} = \frac{u_n - 3}{u_n - 2} \Rightarrow 2^{n+1}(u_n - 2) = u_n - 3$$

$$\Rightarrow 2^{n+1}u_n - 2^{n+2} = u_n - 3$$

$$\Rightarrow 2^{n+1}u_n - u_n = 2^{n+2} - 3$$

$$\Rightarrow u_n(2^{n+1} - 1) = 2^{n+2} - 3 \Rightarrow u_n = \frac{2^{n+2} - 3}{2^{n+1} - 1}$$

$$v_n = 2^{n+1} \xrightarrow{n \rightarrow +\infty} +\infty$$

$(v_n)_{n \in \mathbb{N}}$ Div

$$u_n \xrightarrow{n \rightarrow +\infty} \frac{+\infty}{+\infty} \text{ f.v.}$$

$$u_n = \frac{2^{n+2} - 3}{2^{n+1} - 1} \underset{n \rightarrow +\infty}{\sim} \frac{2^{n+2}}{2^{n+1}} = 2 \rightarrow 2$$

$(u_n)_{n \in \mathbb{N}}$ CV vers 2

$$\left(u_n = \frac{2^{n+2} \left(1 - \frac{3}{2^{n+2}}\right)}{2^{n+1} \left(1 - \frac{1}{2^{n+1}}\right)} = 2 \frac{\left(1 - \frac{3}{2^{n+2}}\right)}{1 - \frac{1}{2^{n+1}}} \rightarrow 2 \times \frac{1}{1} = 2 \right)$$

$$\textcircled{2} \quad u_n = \frac{n!}{2^n}$$

$$\frac{u_{n+1}}{u_n} = \frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}} = \frac{(n+1)!}{2^{n+1}} \times \frac{2^n}{n!} = \frac{2^n (n+1)!}{2^{n+1} n!}$$

$$\frac{2^m}{2^{n+1}} = \frac{2^m}{2^n \times 2} = \frac{1}{2} \quad \text{et} \quad \frac{(n+1)!}{n!} = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n \cdot (n+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n} = n+1$$

donc $\frac{u_{n+1}}{u_n} = \frac{n+1}{2} > \frac{1+1}{2} = 1$ qd $n > 1$

$$\frac{u_{n+1}}{u_n} > 1 \Rightarrow u_{n+1} > u_n, \forall n \geq 1$$

$(u_n)_{n \geq 1}$ est strictement croissante

③ Somme géométrique finie: $1 + r + r^2 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}$ qd $r \neq 1$

avec $r = \frac{1}{2}$

$$1 + \left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} \right) = \frac{\frac{1}{2}^{n+1} - 1}{\frac{1}{2} - 1}$$

$$\sum_{k=1}^n \frac{1}{2^k} = \frac{\frac{1}{2}^{n+1} - 1}{\frac{1}{2} - 1} - 1$$

$$= \frac{2^{-n-1} - 1}{-\frac{1}{2}} - 1 = -2 \cdot (2^{-n-1} - 1) - 1$$

$$= -2^{-n} + 2 - 1 = 1 - 2^{-n} = \sum_{k=1}^n \frac{1}{2^k}$$

$$\underbrace{0}_{\text{minore}} \leq U_n = \sum_{k=1}^n \frac{1}{2^k} \leq \sum_{k=1}^n \frac{1}{2^k} \leq 1 - \frac{1}{2^n} \leq \underbrace{1}_{\text{majorée}} \quad \text{borne}$$

$$u_{n+1} - u_n = \sum_{k=1}^{n+1} \frac{1}{k2^k} - \sum_{k=1}^n \frac{1}{k2^k}$$

$$= \frac{1}{(n+1)2^{n+1}} > 0$$

$$u_{n+1} - u_n > 0 \Rightarrow u_{n+1} > u_n, \forall n$$

donc $(u_n)_{n \geq 0}$ est strict⁺ croissante

$(u_n)_{n \geq 1}$ CV car elle est croissante et majorée

on verra plus tard que $\lim_{n \rightarrow +\infty} u_n = \ln(2)$

④ la suite est arithmétique, donc $\forall n \in \mathbb{N} \quad u_n = n \cdot r + u_0$

$$\begin{cases} u_{100} = 52 = 100r + u_0 \\ u_{200} = 3 = 200r + u_0 \end{cases} \quad \begin{aligned} l_2 - l_1: -49 &= 100r \Rightarrow \underline{r = -0,49} \\ l_2 - 2l_1: 3 - 104 &= -u_0 \Rightarrow \underline{u_0 = 101} \end{aligned}$$

$$u_n = -0,49n + 101, \quad \forall n \in \mathbb{N}$$

⑤ $u_0 = 1$ et $u_{n+1} = u_n + \frac{1}{u_n^2}$ $u_{n+1} - u_n = \frac{1}{u_n^2} > 0$

donc (u_n) est croissante

Si elle CV vers l , alors $l = l + \frac{1}{l^2} \Rightarrow 0 = \frac{1}{l^2}$ impossible

donc elle Div

La suite $(u_n)_{n \in \mathbb{N}}$ est croissante et Div, donc $\lim_{n \rightarrow \infty} u_n = +\infty$

$$\textcircled{1} u_0 = 1 \quad u_{n+1} = \frac{u_n - 6}{u_n - 4} \quad v_n = \frac{u_n - 3}{u_n - 2}$$

$$v_{n+1} = \frac{u_{n+1} - 3}{u_{n+1} - 2} = \frac{\frac{u_n - 6}{u_n - 4} - 3}{\frac{u_n - 6}{u_n - 4} - 2} \quad \begin{array}{l} \times (u_n - 4) \\ \times (u_n - 4) \end{array}$$

$$v_{n+1} = \frac{u_n - 6 - 3(u_n - 4)}{u_n - 6 - 2(u_n - 4)} = \frac{-2u_n - 6 + 12}{-u_n - 6 + 8} = \frac{6 - 2u_n}{2 - u_n} = 2 \frac{(3 - u_n)}{(2 - u_n)}$$

$$= 2 \frac{u_n - 3}{u_n - 2} = 2 v_n$$

$v_{n+1} = 2 v_n$, $\forall n \in \mathbb{N}$. Donc $(v_n)_{n \in \mathbb{N}}$ est une suite géométrique de raison 2

$$v_n = 2^n v_0 = 2^n \cdot \frac{u_0 - 3}{u_0 - 2} = 2^n \cdot \frac{-2}{-1} = 2^{n+1}$$

$$v_n = 2^{n+1}$$

$$v_n = 2^{n+1} = \frac{u_n - 3}{u_n - 2} \Rightarrow 2^{n+1}(u_n - 2) = u_n - 3$$

$$\Rightarrow 2^{n+1}u_n - 2^{n+2} = u_n - 3$$

$$\Rightarrow 2^{n+1}u_n - u_n = 2^{n+2} - 3$$

$$\Rightarrow u_n(2^{n+1} - 1) = 2^{n+2} - 3$$

$$\Rightarrow u_n = \frac{2^{n+2} - 3}{2^{n+1} - 1}$$

$\xrightarrow{+\infty}$
 $\xrightarrow{+\infty}$
 F.I

$$v_n = 2^{n+1} \xrightarrow{n \rightarrow +\infty} +\infty$$

$(v_n)_{n \in \mathbb{N}}$ D.I.V

$$u_n = \frac{2^{n+2} \left(4 - \frac{3}{2^n}\right)}{2^{n+1} \left(2 - \frac{1}{2^n}\right)} \rightarrow \frac{4}{2} = 2$$

$(u_n)_{n \in \mathbb{N}}$ C.V vers 2

$$\textcircled{2} \quad u_n = \frac{n!}{2^n}$$

$$\frac{u_{n+1}}{u_n} = \frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}} = \frac{(n+1)!}{2^{n+1}} \times \frac{2^n}{n!} = \frac{2^n}{2^{n+1}} \frac{(n+1)!}{n!}$$

$$\frac{2^n}{2^{n+1}} = \frac{2^n}{2^n \times 2} = \frac{1}{2} \quad \text{et} \quad \frac{(n+1)!}{n!} = \frac{1 \cdot 2 \cdot \dots \cdot n \cdot (n+1)}{1 \cdot 2 \cdot \dots \cdot n} = n+1$$

$$\text{donc} \quad \frac{u_{n+1}}{u_n} = \frac{n+1}{2} > \frac{1+1}{2} = 1 \quad \text{qd } n > 1$$

$$\frac{u_{n+1}}{u_n} > 1 \Rightarrow u_{n+1} > u_n, \quad \forall n \geq 1$$

$(u_n)_{n \in \mathbb{N}}$ est strictement croissante

$$(3) \sum_{k=1}^n \frac{1}{2^k} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots + \left(\frac{1}{2}\right)^n$$

$$(1) 1 + r + r^2 + \dots + r^n = \frac{1-r^{n+1}}{1-r} \quad \text{qd } r \neq 1$$

avec $r = \frac{1}{2}$:

$$(1) 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^n = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{2^{n+1}}}{\frac{1}{2}} = 2 \times \left(1 - \frac{1}{2^{n+1}}\right)$$

$$= 2 - \frac{1}{2^n}$$

On en déduit :

$$\sum_{k=1}^n \frac{1}{2^k} = 1 - \frac{1}{2^n}$$

$$0 \leq u_n = \sum_{k=1}^n \frac{1}{k 2^k} \leq \sum_{k=1}^n \frac{1}{2^k} = 1 - \frac{1}{2^n} \leq 1$$

↑ minorant
↑ majorant

donc la suite $(u_n)_{n \in \mathbb{N}}$ est bornée

$$u_{n+1} - u_n = \sum_{k=1}^{n+1} \frac{1}{k2^k} - \sum_{k=1}^n \frac{1}{k2^k} = \dots$$

$$= \frac{1}{(n+1)2^{n+1}} > 0$$

$$u_{n+1} - u_n > 0 \Rightarrow u_{n+1} > u_n, \forall n$$

donc $(u_n)_{n \geq 0}$ est strict⁺ croissante

$(u_n)_{n \geq 1}$ CV car elle est croissante et majorée

④ La suite est arithmétique, donc $\forall n \in \mathbb{N}, u_n = n r + u_0$

$$u_{100} = 52 = 100r + u_0$$

$$l_2 - l_1: -49 = 100r \Rightarrow r = \underline{-0,49}$$

$$u_{200} = 3 = 200r + u_0$$

$$l_2 - 2l_1: 3 - 104 = -u_0 \Rightarrow u_0 = \underline{101}$$

$$u_n = -0,49n + 101, \forall n \in \mathbb{N}$$

⑤ $u_0 = 1$

et

$$u_{n+1} = u_n + \frac{1}{u_n^2}$$

$$u_{n+1} - u_n = \frac{1}{u_n^2} > 0$$

donc $(u_n)_n$ est croissante

Si la suite (u_n) CV vers l ,

$$\text{alors } l = l + \frac{1}{l^2} \Rightarrow 0 = \frac{1}{l^2} \Rightarrow 0 = 1$$

contradiction. Donc la suite (u_n) Div

La suite (u_n) est croissante et Div, donc $\lim_{n \rightarrow +\infty} u_n = +\infty$

$$(6) \quad C_n = \sum_{k=0}^n \cos(k\theta) \quad \text{et} \quad S_n = \sum_{k=0}^n \sin(k\theta)$$

$$\begin{aligned} C_n + iS_n &= \sum_{k=0}^n (\cos(k\theta) + i \sin(k\theta)) = \sum_{k=0}^n e^{ik\theta} \\ &= \sum_{k=0}^n (e^{i\theta})^k = \frac{(e^{i\theta})^{n+1} - 1}{e^{i\theta} - 1} = \frac{e^{i\theta(n+1)} - 1}{e^{i\theta} - 1} \\ &= \frac{e^{i\frac{\theta}{2}(n+1)} \left[2i \sin \frac{\theta(n+1)}{2} \right]}{e^{i\frac{\theta}{2}} \left[e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} \right]} = e^{i\frac{\theta}{2}n} \frac{\sin \frac{\theta(n+1)}{2}}{\sin \left(\frac{\theta}{2} \right)} \end{aligned}$$