

red

3c

$$\frac{1}{1-3x+2x^2} = \frac{1}{(1-x)(1-2x)} = \frac{2}{1-2x} - \frac{1}{1-x}$$

$$= 2 \sum_0^{+\infty} (2x)^n - \sum_0^{+\infty} x^n \quad \text{pour } x \in]-\frac{1}{2}, \frac{1}{2}[$$

$|2x| < 1 \quad |x| < 1$

$$= \sum_0^{+\infty} (2 \cdot 2^n - 1) x^n = \sum_{n=0}^{+\infty} (2^{n+1} - 1) x^n$$

Calcul du rayon : $2^{-n+1} \sim 2^{n+1} = a_n$.

donc $\sum_0^{+\infty} (2^{n+1} - 1) x^n$ et $\sum_0^{+\infty} 2^{n+1} x^n$ ont \hat{m} rayon

$$\left| \frac{u_{n+1}}{u_n} \right| = \frac{2^{n+2}}{2^{n+1}} \frac{|x|^{n+1}}{|x|^n} = 2|x|$$

Si $2|x| < 1$ alors $\sum u_n \text{ CV}$
 Si $2|x| > 1$ ———— D.V

$R = \frac{1}{2}$

$$\frac{1}{1-3x+2x^2} = \sum_0^{+\infty} (2^{n+1} - 1) x^n, \quad \forall x \in]-\frac{1}{2}, \frac{1}{2}[$$

$$\frac{1}{2-x} = \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2} \sum_0^{+\infty} \left(\frac{x}{2}\right)^n = \sum_0^{+\infty} \frac{1}{2^{n+1}} x^n$$

pour $\frac{x}{2} \in]-1, 1[$, donc pour $x \in]-2, 2[$

$$\textcircled{4} 4xy'' + 2y' + y = 0$$

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n$$

$$f'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$$

$$f''(x) = \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2}$$

$$4x f''(x) + 2f'(x) + f(x) = 4x \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=1}^{+\infty} n a_n x^{n-1} + \sum_{n=0}^{+\infty} a_n x^n = 0$$

$$= \sum_{n=2}^{+\infty} 4n(n-1) a_n x^{n-1} + \sum_{n=1}^{+\infty} 2n a_n x^{n-1} + \sum_{n=0}^{+\infty} a_n x^n = 0$$

$$= \sum_{n=0}^{+\infty} 4(n+1)n a_{n+1} x^n + \sum_{n=0}^{+\infty} 2(n+1) a_{n+1} x^n + \sum_{n=0}^{+\infty} a_n x^n = 0$$

$$= \sum_{n=0}^{+\infty} [4(n+1)n a_{n+1} + 2(n+1) a_{n+1} + a_n] x^n = 0 = \sum_{n=0}^{+\infty} 0 x^n$$

$$\forall n \geq 0, 4(n+1)a_{n+1} + 2(n+1)a_{n+1} + a_n = 0$$

on décale de 1 :

$$\forall n \geq 1, 4n(n-1)a_n + 2na_n + a_{n-1} = 0$$

$$2n(2(n-1)a_n + a_{n-1}) = 0$$

$$\forall n \geq 1, 2n(2n-1)a_n + a_{n-1} = 0$$

$$n=1: 2 \cdot 1 \cdot a_1 + a_0 = 0 \Rightarrow a_1 = -\frac{a_0}{2}$$

$$n=2: 4 \cdot 3 a_2 + a_1 = 0 \Rightarrow a_2 = -\frac{a_1}{3 \cdot 5} = \frac{a_0}{2 \cdot 3 \cdot 4} = \frac{a_0}{4!}$$

$$n=3: 6 \cdot 5 a_3 + a_2 = 0 \Rightarrow a_3 = \frac{-a_2}{5 \cdot 6} = \frac{-a_0}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = -\frac{a_0}{6!}$$

Il semble que $a_n = \frac{(-1)^n}{(2n)!} a_0$

Si $a_0 = 1$: $f(x) = \sum_0^{+\infty} \frac{(-1)^n}{(2n)!} x^n = \sum_0^{+\infty} \frac{(-1)^n}{(2n)!} (\sqrt{x})^{2n}$ qd $x \geq 0$ car $(\sqrt{x})^2 = x$
 $= \cos(\sqrt{x})$

$$\text{qd } x \leq 0: f(x) = \sum_0^{+\infty} \frac{(-x)^n}{(2n)!} = \sum_0^{+\infty} \frac{(\sqrt{-x})^{2n}}{(2n)!} = \underline{\underline{\text{ch}(\sqrt{-x})}}$$

ANS 2011 Ex 3 partie B

$$x y''(x) + y'(x) + \omega^2 x y(x) = 0 \quad \omega > 0$$

$$\text{en } \underline{x=0}: 0 + y'(0) + 0 = 0 \Rightarrow \underline{y'(0)=0}$$

$$y = \sum_0^{+\infty} a_n x^n \quad y' = \sum_1^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 \dots$$

$$y'(0) = \underline{a_1} = 0$$

$$y''(x) = \sum_1^{\infty} a_n n(n-1) x^{n-2}$$

$$\begin{aligned} x y'' + y' + \omega^2 x y &= \sum_1^{\infty} a_n n(n-1) x^{n-1} + \sum_1^{\infty} n a_n x^{n-1} + \sum_0^{\infty} \omega^2 a_n x^{n+1} \\ &= \sum_0^{+\infty} a_{n+2} (n+1)n x^n + \sum_0^{\infty} (n+1) a_{n+1} x^n + \sum_1^{\infty} \omega^2 a_{n-1} x^n \\ &= \sum_1^{\infty} a_{n+2} (n+1)n x^n + \sum_1^{\infty} (n+1) a_{n+1} x^n + \sum_1^{\infty} \omega^2 a_{n-1} x^n = 0 \end{aligned}$$

$$\sum_1^{\infty} [a_{n+1} (n+1)n + (n+1)a_{n+1} + \omega^2 a_{n-1}] x^n = 0$$

D'après l'unicité du développement en Series Entières :

$$\forall n \geq 1 \quad (n+1)^2 a_{n+1} + \omega^2 a_{n-1} = 0 \Rightarrow a_{n+1} = \frac{-\omega^2 a_{n-1}}{(n+1)^2}$$

$$\Rightarrow \forall n \geq 0 \quad a_{n+2} = \frac{-\omega^2 a_n}{(n+2)^2}$$

Notons par récurrence que $a_{2n} = \frac{(-1)^n \omega^{2n}}{2^{2n} n!^2} a_0$

$$\begin{aligned}
 \textcircled{3c} \quad \frac{1}{1-3x+2x^2} &= \frac{1}{(1-x)(1-2x)} = \frac{-1}{1-x} + \frac{2}{1-2x} \\
 &= -\sum_{n=0}^{+\infty} x^n + 2 \sum_{n=0}^{+\infty} (2x)^n \\
 &= -\sum_{n=0}^{+\infty} x^n + 2 \sum_{n=0}^{+\infty} 2^n x^n \\
 &= \sum_{n=0}^{+\infty} [2^{n+1} - 1] x^n
 \end{aligned}$$

pour $x \in]-1, 1[$
 pour $2x \in]-1, 1[$
 $x \in]-\frac{1}{2}, \frac{1}{2}[$

$$\forall x \in]-\frac{1}{2}, \frac{1}{2}[\quad R = \frac{1}{2}$$

$$\begin{aligned}
 \text{S. E de } f(x) = \frac{1}{2-x} &= \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2} \sum_{n=0}^{+\infty} \left(\frac{x}{2}\right)^n \\
 &= \sum_{n=0}^{+\infty} \frac{x^n}{2^{n+1}}
 \end{aligned}$$

$\forall x \in]-2, 2[$
 (car $\frac{x}{2} \in]-1, 1[$)

$$\textcircled{4} \quad 4xy'' + 2y' + y = 0 \quad f(x) = \sum_0^{+\infty} a_n x^n \quad f'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$$

$$f''(x) = \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2}$$

$$4xy'' + 2y' + y = \sum_1^{\infty} 4n(n-1)a_n x^{n-1} + \sum_1^{\infty} 2na_n x^{n-1} + \sum_0^{+\infty} a_n x^n$$

$$= \sum_0^{\infty} 4(n+1)na_{n+1} x^n + \sum_0^{\infty} 2(n+1)a_{n+1} x^n + \sum_0^{+\infty} a_n x^n$$

$$= \sum_0^{\infty} \left[4(n+1)na_{n+1} + 2(n+1)a_{n+1} + a_n \right] x^n$$

$$= \sum_0^{\infty} \left[2(n+1)(2n+1)a_{n+1} + a_n \right] x^n = 0 = \sum 0x^n$$

D'après l'unicité des D&E en S.E: $\forall n \geq 0 \quad 2(n+1)(2n+1)a_{n+1} + a_n = 0$

ou de la forme:

$$\forall n \geq 1 \quad 2n(2n-1)a_n + a_{n-1} = 0$$

$$a_0 = 1$$

$$\text{pour } n=1: \quad 2a_1 + a_0 = 0$$

$$\text{pour } n=2: \quad 4 \cdot 3 a_2 + a_1 = 0$$

$$\text{pour } n=3: \quad 6 \cdot 5 a_3 + a_2 = 0$$

$$a_1 = -\frac{1}{2} = \frac{(-1)^1}{2!}$$

$$a_2 = -\frac{a_1}{2 \cdot 3} = \frac{(-1)^2}{2 \cdot 3 \cdot 4} = \frac{(-1)^2}{4!}$$

$$a_3 = -\frac{a_2}{5 \cdot 6} = \frac{-1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = \frac{(-1)^3}{6!}$$

Il semble que $a_n = \frac{(-1)^n}{(2n)!}$ Récurrence :

Initialisation : pour $n=0$

Hérédité : Je prends un $n \geq 0$. Je suppose que $a_n = \frac{(-1)^n}{(2n)!}$

$$a_{n+1} = \frac{-a_n}{(2n+2)(2n+1)} = \frac{-\frac{(-1)^n}{(2n)!}}{(2n+2)(2n+1)} = \frac{-(-1)^n}{(2n)! \cdot (2n+1)(2n+2)} = \frac{(-1)^{n+1}}{(2n+2)!} \text{ YES!}$$

Conclusion : $\forall n \geq 0 \quad a_n = \frac{(-1)^n}{(2n)!}$

$$f(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

si $x > 0$: $\cos(\sqrt{x}) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} (\sqrt{x})^{2n}$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^n = f(x)$$

si $x \leq 0$: $\cosh(\sqrt{-x}) = \sum_{n=0}^{+\infty} \frac{1}{(2n)!} (\sqrt{-x})^{2n} = \sum_{n=0}^{+\infty} \frac{(-x)^n}{(2n)!} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^n}{(2n)!} = f(x)$

ATS 2021 exo 3 partie B

$$x y'' + y' + \omega^2 x y = 0$$

① Montrer que $y'(0) = 0$

en $x=0$: $0 + y'(0) + 0 = 0 \Rightarrow \underline{y'(0) = 0}$

② $y(x) = \sum_{n=0}^{+\infty} a_n x^n$ Que vaut a_1 ?

On dérive : $y'(x) = \sum_{n=0}^{+\infty} n a_n x^{n-1} = 0 + a_1 + 2a_2 x + 3a_3 x^2 + \dots$

$$y'(0) = \underline{a_1 = 0}$$

③ Montrer que $\forall n \geq 0 \quad a_{n+2} = \frac{-\omega^2 a_n}{(n+2)^2}$

$$x y'' + y' + \omega^2 x y = \sum_{n=0,1,2}^{+\infty} n(n-1) a_n x^{n-1} + \sum_{n=0,1}^{+\infty} n a_n x^{n-1} + \sum_0^{+\infty} \omega^2 a_n x^{n+1}$$

$$\sum_{n=1}^{+\infty} (n+1)n a_{n+1} x^n + \sum_0^{+\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{+\infty} \omega^2 a_{n-1} x^n = 0$$

$\uparrow a_1 = 0$

$$\sum_{n=1}^{+\infty} \left[(n+1)na_{n+1} + (n+1)a_{n+1} + \omega^2 a_{n-1} \right] x^n = 0 = \sum_1^{+\infty} 0 x^n$$

$= 0$ (unicité des coeff en S. E.)

$$(n+1)a_{n+1}(n+1) + \omega^2 a_{n-1} = 0$$

$$\forall n \geq 1: (n+1)^2 a_{n+1} + \omega^2 a_{n-1} = 0$$

on décale de 1

$$\forall n \geq 0: (n+2)^2 a_{n+2} + \omega^2 a_n = 0$$

CQFD

$$a_{n+2} = \frac{-\omega^2 a_n}{(n+2)^2}$$

M que $a_{2n} = \frac{(-1)^n \omega^{2n}}{2^{2n} (n!)^2} a_0$ par récurrence