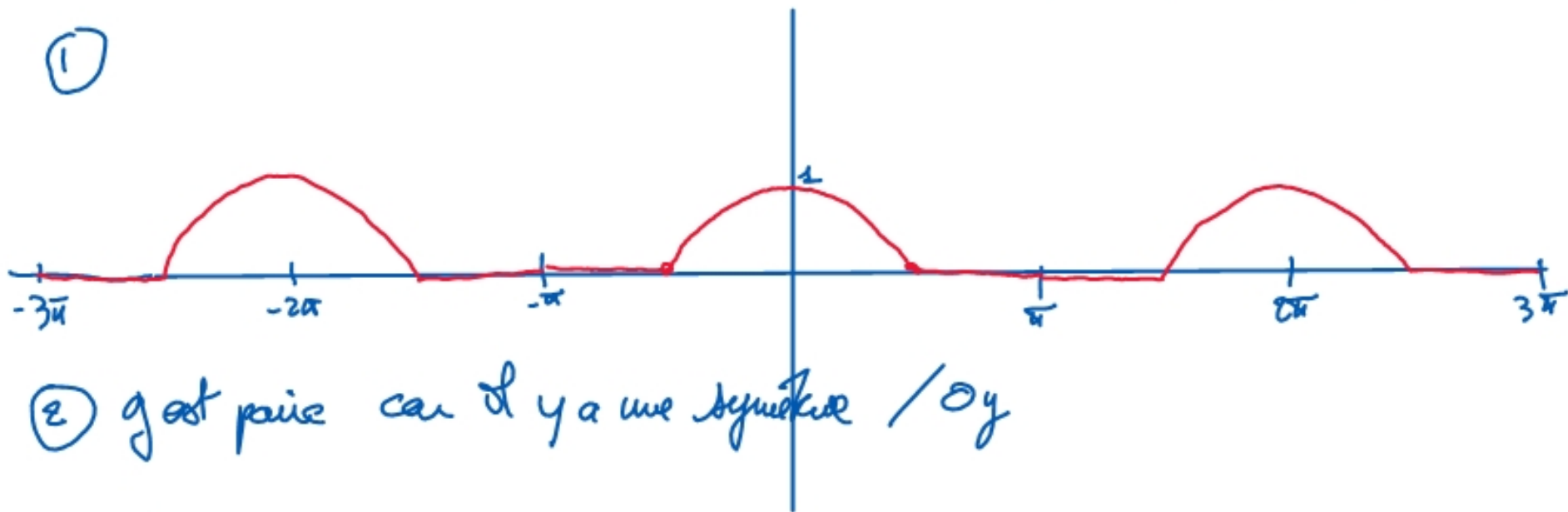


seee

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$$g(t) = \begin{cases} \cos t \text{ sur } [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 0 \text{ sur } ]-\pi, -\frac{\pi}{2}] \cup ]\frac{\pi}{2}, \pi] \end{cases}$$

①



②  $g$  est paire car il y a une symétrie /  $Oy$

③.  $b_n = 0$  car  $g$  est paire

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) dt = \frac{1}{\pi} \int_0^{\pi} g(t) dt = \frac{1}{\pi} \int_0^{\pi/2} \cos(t) dt = \frac{1}{\pi} [\sin t]_0^{\pi/2} = \frac{1}{\pi} = a_0$$

$$a_1 = \frac{2}{2\pi} \int_{-\pi}^{\pi} g(t) \cos t dt = \frac{2}{\pi} \int_0^{\pi} g(t) \cos t dt = \frac{2}{\pi} \int_0^{\pi/2} \cos^2 t dt$$

$$\cos(2t) = 2\cos^2 t - 1 \Rightarrow \cos^2 t = \frac{\cos(2t) + 1}{2} \quad a_1 = \frac{1}{\pi} \int_0^{\pi/2} (\cos(2t) + 1) dt = \frac{1}{\pi} \left[ \frac{\sin(2t)}{2} + t \right]_0^{\pi/2} = \frac{1}{2}$$

$$a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} = \frac{1}{\pi} \int_0^{\pi} = \frac{2}{\pi} \int_0^{\pi/2} \cos t \cos(nt) dt \quad \text{Ild} \left| \begin{array}{l} u = \cos t \\ u' = -\sin t \\ v = \frac{\sin(nt)}{n} \end{array} \right.$$

$$a_n = \frac{2}{\pi} \left( \left[ \cancel{\cos t \frac{\sin(nt)}{n}} \right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\sin t \sin(nt)}{n} dt \right) = \frac{2}{\pi n} \int_0^{\pi/2} \sin t \sin nt dt \quad \left| \begin{array}{l} u = \sin t \quad u' = \cos t \\ v = \frac{\sin(nt)}{n} \quad v' = \cos(nt) \end{array} \right.$$

$$a_n = \frac{2}{\pi n} \left( \left[ -\frac{\sin t \cos(nt)}{n} \right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos t \cdot \cos(nt)}{n} dt \right)$$

$$a_n = \frac{2}{\pi n} \left( -\frac{\cos(n\pi/2)}{n} + \frac{1}{n} \int_0^{\pi/2} \cos t \cos nt dt \right)$$

$$\parallel \frac{\pi^2}{2} a_n \quad \frac{n^2-1}{n^2}$$

$$a_n = \frac{-2}{\pi n^2} \cos(n\pi/2) + \frac{1}{n^2} a_n \Rightarrow a_n \left( 1 - \frac{1}{n^2} \right) = -\frac{2}{\pi n^2} \cos(n\pi/2)$$

$$a_n = \frac{-2 \cos(n\pi/2)}{\pi(n^2-1)}$$

Si  $n$  est pair:  $n=2p$   $a_n = a_{2p} = -\frac{2\cos(p\pi)}{\pi(4p^2-1)} = \frac{-2(-1)^p}{\pi(4p^2-1)} = \frac{+2(-1)^{p+1}}{\pi(4p^2-1)}$

Si  $n$  est impair:  $n=2p+1$   $\cos(n\frac{\pi}{2}) = \cos(2p\frac{\pi}{2} + \frac{\pi}{2})$   
 $= \cos(p\pi + \frac{\pi}{2})$

$= \underbrace{\cos(p\pi)}_{(-1)^p} \underbrace{\cos(\frac{\pi}{2})}_0 - \underbrace{\sin(p\pi)}_0 \underbrace{\sin(\frac{\pi}{2})}_1 = 0$

④  $g$  est de classe  $C'$  par morceaux et continue, donc  $S_g(x) = g(x)$  (DIRICHLET)

si  $x \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$

$S_g(x) = \frac{1}{\pi} + \frac{1}{2} \cos(x) + \sum_{p=1}^{+\infty} \frac{2(-1)^{p+1}}{\pi(4p^2-1)} \cos(2px) = \cos x$

$a_0$

$a_1$

$S_g(x)$

$g(x)$

$\frac{1}{\pi} + \frac{2}{\pi} \sum_{p=1}^{+\infty} \frac{(-1)^{p+1}}{4p^2-1} \cos(2px) = \cos x - \frac{1}{2} \cos x = \frac{1}{2} \cos x$

$$\cos(t) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{p=1}^{+\infty} \frac{(-1)^{p+1}}{4p^2-1} \cos(2pt) \quad \forall t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

On a:  $\cos(0)=1 = \frac{2}{\pi} + \frac{4}{\pi} \sum_{p=1}^{+\infty} \frac{(-1)^{p+1}}{4p^2-1} \Rightarrow \frac{4}{\pi} \sum_{p=1}^{+\infty} \frac{(-1)^{p+1}}{4p^2-1} = 1 - \frac{2}{\pi} = \frac{\pi-2}{\pi}$

$$\Rightarrow \boxed{\sum_{p=1}^{+\infty} \frac{(-1)^{p+1}}{4p^2-1} = \frac{\pi-2}{4}}$$

(5)  $g$  est continue par morceaux. On peut appliquer Parseval

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f^2(t) dt = \frac{1}{\pi} \int_0^{\pi} f^2(t) dt = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \cos^2(t) dt = a_0^2 + \frac{1}{2} \sum_{p=1}^{\infty} (a_p^2 + b_p^2)$$

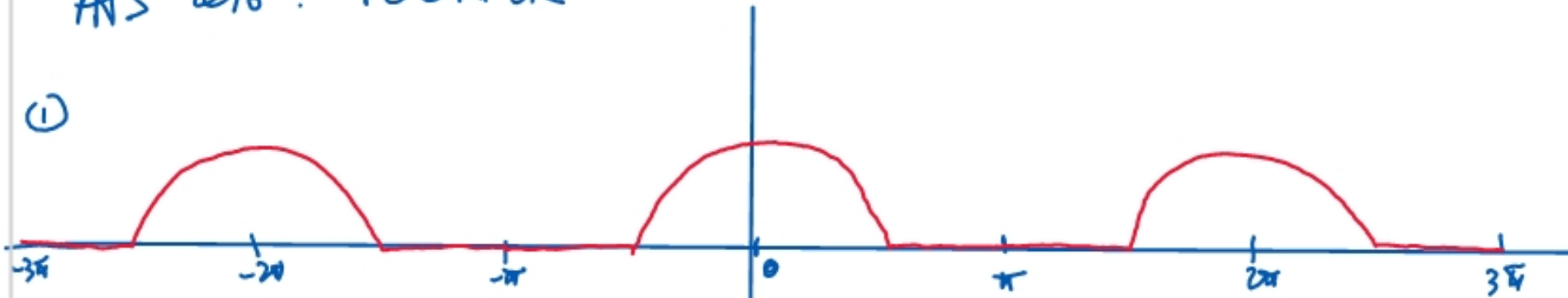
$$\cos^2 t = \frac{\cos(2t)+1}{2} \quad \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} (\cos(2t)+1) dt = \frac{1}{2\pi} \cdot \frac{\pi}{2} = \frac{1}{4} = \frac{1}{\pi^2} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} \sum_{p=1}^{+\infty} \frac{4}{\pi^2 (4p^2-1)^2}$$

$$\frac{1}{4} - \frac{1}{\pi^2} - \frac{1}{8} = \frac{2}{\pi^2} \sum_{p=1}^{\infty} \frac{1}{(4p^2-1)^2}$$

$$\frac{\pi^2-8}{8\pi^2} = \frac{2}{\pi^2} \sum$$

$$\Rightarrow \boxed{\sum_{p=1}^{\infty} \frac{1}{(4p^2-1)^2} = \frac{\pi^2-8}{16}}$$

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②  $g$  est paire, car il y a une symétrie /Oy

③  $b_n = 0$  car  $g$  est paire

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) dt = \frac{1}{\pi} \int_0^{\pi} g(t) dt = \frac{1}{\pi} \int_0^{\pi/2} \cos t dt = \frac{1}{\pi} [\sin t]_0^{\pi/2} = \frac{1}{\pi} \quad a_0 = \frac{1}{\pi}$$

$$a_1 = \frac{2}{2\pi} \int_{-\pi}^{\pi} g(t) \cos t dt = \frac{4}{2\pi} \int_0^{\pi} g(t) \cos t dt = \frac{2}{\pi} \int_0^{\pi/2} \cos^2 t dt$$

$$\cos(2t) = 2\cos^2 t - 1 \Rightarrow \cos^2 t = \frac{\cos(2t) + 1}{2}$$

$$a_1 = \frac{2}{\pi} \int_0^{\pi/2} (\cos(2t) + 1) dt$$

$$a_1 = \frac{1}{\pi} \left[ \frac{\sin(2t)}{2} + t \right]_0^{\pi/2} = \frac{1}{\pi} \frac{\pi}{2} = \frac{1}{2}$$

$$a_1 = \frac{1}{2}$$

$$a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} \dots = \frac{4}{2\pi} \int_0^{\pi} \dots = \frac{2}{\pi} \int_0^{\pi/2} \cos t \cdot \cos nt \, dt$$

On linéarise :  $\cos t = \frac{e^{it} + e^{-it}}{2}$        $\cos nt = \frac{e^{int} + e^{-int}}{2}$

$$\begin{aligned} \cos t \cdot \cos nt &= \frac{1}{4} (e^{it} + e^{-it})(e^{int} + e^{-int}) = \frac{1}{4} (\underbrace{e^{i(n+1)t}} + \underbrace{e^{i(1-n)t}} + \underbrace{e^{-i(1-n)t}} + \underbrace{e^{-i(n+1)t}}) \\ &= \frac{1}{4} (2 \cos((n+1)t) + 2 \cos((1-n)t)) = \frac{1}{2} (\cos \dots + \cos \dots) \end{aligned}$$

$$a_n = \frac{2}{\pi} \frac{1}{2} \left[ \frac{\sin((n+1)t)}{n+1} + \frac{\sin(1-n)t}{1-n} \right]_0^{\pi/2} = \frac{1}{\pi(n+1)} \sin\left(\frac{(n+1)\pi}{2}\right) + \frac{1}{\pi(n-1)} \sin\left(\frac{(n-1)\pi}{2}\right)$$

Si n est impair :  $n = 2p+1$        $\sin\left(\frac{2p+2}{2}\pi\right) = \sin(p+1)\pi = 0$

$$\sin\left(\frac{2p}{2}\pi\right) = \sin(p\pi) = 0$$

donc  $a_n = 0$

Si n est pair  $n = 2p$

$$a_{2p} = \frac{1}{\pi(2p+1)} \underbrace{\sin\left(p\pi + \frac{\pi}{2}\right)}_{\substack{\sin p\pi / \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \cos(p\pi) \\ \text{"0" } \quad \quad \quad (-1)^p}} + \frac{1}{\pi(2p-1)} \underbrace{\sin\left(p\pi - \frac{\pi}{2}\right)}_{\substack{\sin p\pi / \cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) \cos(p\pi) \\ \text{"0" } \quad \quad \quad -(-1)^p}}$$

$$\begin{aligned} a_{2p} &= \frac{(-1)^p}{\pi(2p+1)} + \frac{(-1)^{p+1}}{\pi(2p-1)} = \frac{(-1)^p}{\pi} \left( \frac{1}{2p+1} - \frac{1}{2p-1} \right) \\ &= \frac{(-1)^p}{\pi} \frac{2p-1 - (2p+1)}{(2p+1)(2p-1)} = \frac{-(-1)^p 2}{\pi(4p^2-1)} = \frac{2(-1)^{p+1}}{(4p^2-1)\pi} \end{aligned}$$

④  $g$  est de classe  $C^1$  par morceaux et continue, donc  $S_g(x) = g(x), \forall x$

$$S_g(x) = a_0 + \sum_{n=1}^{+\infty} a_n \cos(nx) = \frac{1}{\pi} + \frac{1}{2} \cos x + \sum_{p=1}^{+\infty} a_{2p} \cos(2px) = \text{DIRICHLET } g(x) = \cos x \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\begin{aligned} \cos x - \frac{1}{2} \cos x &= \frac{1}{\pi} + \sum_{p=1}^{+\infty} \frac{2(-1)^{p+1}}{\pi(4p^2-1)} \cos(2px) \\ \frac{1}{2} \cos x &= \quad \quad \quad \Rightarrow \cos t = \frac{2}{\pi} + \frac{4}{\pi} \sum_{p=1}^{+\infty} \frac{(-1)^{p+1}}{4p^2-1} \cos(2px) \end{aligned}$$

$$\underline{\text{ent } = 0} : 1 = \frac{2}{\pi} + \frac{4}{\pi} \sum_1^{\infty} \frac{(-1)^{p+1}}{4p^2-1}$$

$$\Rightarrow 1 - \frac{2}{\pi} = \frac{\pi-2}{\pi} = \frac{4}{\pi} \sum \Rightarrow \sum_1^{\infty} \frac{(-1)^{p+1}}{4p^2-1} = \frac{\pi-2}{4}$$