

①

$$f(\vec{u}) = a\vec{u}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f(\vec{u}) = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 & -1 & 1 \\ 2 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} = 4\vec{u}$$

$f(\vec{u}) = 4\vec{u}$, donc \vec{u} est un vecteur propre pour la valeur propre 4

③

$$\Rightarrow \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix}}_A \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \quad \boxed{b} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = A^n \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}$$

suite géométrique

$$\boxed{c} \begin{vmatrix} 2-x & -1 & -1 \\ -1 & 2-x & -1 \\ 1 & 1 & 4-x \end{vmatrix} \stackrel{4-L_2}{=} \begin{vmatrix} 3-x & -3+x & 0 \\ -1 & 2-x & -1 \\ 1 & 1 & 4-x \end{vmatrix} = \begin{vmatrix} 3-x & 0 & 0 \\ -1 & 1-x & -1 \\ 1 & 2 & 4-x \end{vmatrix}$$

$$\stackrel{\text{dév}}{=} \begin{vmatrix} 3-x & 1-x & -1 \\ 2 & 4-x & 2 \end{vmatrix} \stackrel{L_1+L_2}{=} \begin{vmatrix} 3-x & 3-x & 3-x \\ 2 & 4-x & 2 \end{vmatrix} \stackrel{L_1-L_2}{=} \begin{vmatrix} 3-x & 0 & 0 \\ 2 & 2-x & 2 \end{vmatrix} \stackrel{c_1+c_2}{=} \underline{\underline{(3-x)^2(2-x)}}$$

J'appelle $\mathcal{B} = (\vec{i}, \vec{j}, \vec{k})$ la base canonique de \mathbb{R}^3

J'appelle f l'endomorphisme de \mathbb{R}^3 tel que $\text{Mat}_{\mathcal{B}}(f) = A$

Calcul de $E_3 = \{ \vec{u} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 / f(\vec{u}) = 3\vec{u} \}$.

$$f(\vec{u}) = 3\vec{u} \Leftrightarrow A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix} \Leftrightarrow \begin{cases} 2x - y - z = 3x \\ -x + 2y - z = 3y \\ x + y + 4z = 3z \end{cases}$$

$$\Leftrightarrow \begin{cases} -x - y - z = 0 \\ -x - y - z = 0 \\ x + y + z = 0 \end{cases} \Leftrightarrow x + y + z = 0$$

3 inconnues et 1 eq. $\dim(E_3) = 2$
c'est un plan

$\vec{u} = (1; 0; -1)$ $\vec{v} = (1; -1; 0)$ $f(\vec{u}) = 3\vec{u}$ et $f(\vec{v}) = 3\vec{v}$
Sci, la dimension de l'espace propre E_3 est égale à la multiplicité de 3. Donc A est diagonalisable

Calcul de $E_2 = \{ \vec{w} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 / f(\vec{w}) = 2\vec{w} \}$

$$f(\vec{w}) = 2\vec{w} \Leftrightarrow \begin{cases} 2x - y - z = 2x \\ -x + 2y - z = 2y \\ x + y + 4z = 2y \end{cases} \Leftrightarrow \begin{cases} -y - z = 0 \\ -x - z = 0 \\ x + y + 4z = 0 \end{cases} \Leftrightarrow \begin{cases} -x - z = 0 \\ -y - z = 0 \\ y + z = 0 \end{cases} \Leftrightarrow \begin{cases} x + z = 0 \\ y + z = 0 \end{cases}$$

3 inconnues et 2 eq: choisir 1. Avec $y = t$: $z = -t$ et $x = t$ $\vec{w} = (1; 1; -1)$ convient

$$B' = (\vec{u}, \vec{v}, \vec{w}) \text{ et } \text{Mat}_{B'}^B(f) = \begin{pmatrix} f(\vec{u}) & f(\vec{v}) & f(\vec{w}) \\ \text{dans } B' & \text{dans } B' & \text{dans } B' \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} = A'$$

$$A'^n = \begin{pmatrix} 3^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}$$

$$A^n = P A'^n P^{-1} \text{ où } P \text{ est la matrice de passage de } B \text{ à } B'$$

$$P = \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \\ \text{dans } B & \text{dans } B & \text{dans } B \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \text{ On calcule } P^{-1}$$

$$\begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

Dans P , on a $\vec{u}, \vec{v}, \vec{w}$ en fonction de $\vec{i}, \vec{j}, \vec{k}$

Dans P^{-1} , on a $\vec{i}, \vec{j}, \vec{k}$ en fonction de $\vec{u}, \vec{v}, \vec{w}$

$$\begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \xrightarrow{\vec{u} \quad \vec{v} \quad \vec{w}} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{matrix} \vec{u} - \vec{v} \\ \vec{v} \\ \vec{v} - \vec{w} \end{matrix}$$

$$P^{-1} = \begin{pmatrix} -1 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

colonne 3: $-\vec{j} = \vec{u} - \vec{w} \Rightarrow \vec{w} = \vec{u} - \vec{j} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

colonne 2: $\vec{v} - \vec{j} = \vec{v} \Rightarrow \vec{v} = \vec{j} + \vec{v} = \vec{u} + \vec{v} + \vec{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

colonne 1: $\vec{i} - \vec{k} = \vec{u} \Rightarrow \vec{k} = \vec{i} - \vec{u} = -\vec{u} + \vec{i} + \vec{w}$

$$A^n = P A^{-1} P^{-1} = P \begin{pmatrix} 3^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} -1 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = P \begin{pmatrix} -3^n & -3^n & -2 \cdot 3^n \\ 3^n & 0 & 3^n \\ 2^n & 2^n & 2^n \end{pmatrix}$$

$$A^n = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3^n & -3^n & -2 \cdot 3^n \\ 3^n & 0 & 3^n \\ 2^n & 2^n & 2^n \end{pmatrix} = \begin{pmatrix} 2^n & 2^n - 3^n & 2^n - 3^n \\ 2^n \cdot 3^n & 2^n & 2^n - 3^n \\ 3^n - 2^n & 3^n - 2^n & 2 \cdot 3^n - 2^n \end{pmatrix}$$

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = A^n \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} =$$

$$\begin{pmatrix} a_0 2^n + b_0 (2^n - 3^n) + c_0 (2^n - 3^n) \\ a_0 (2^n \cdot 3^n) + b_0 2^n + c_0 (2^n - 3^n) \\ a_0 (3^n - 2^n) + b_0 (3^n - 2^n) + c_0 (2 \cdot 3^n - 2^n) \end{pmatrix}$$

a_n (points to the first row)
 b_n (points to the second row)
 c_n (points to the third row)

$$\textcircled{4} \quad \begin{vmatrix} -x & -1 & -2 \\ -1 & 1-x & -1 \\ 2 & 1 & 4-x \end{vmatrix} = \begin{vmatrix} -x & -1 & x-2 \\ -1 & 1-x & 0 \\ 2 & 1 & 2-x \end{vmatrix} \stackrel{\substack{C_3 \leftarrow C_3 - C_1 \\ l_3 \leftarrow l_1 + l_3}}{=} \begin{vmatrix} -x & -1 & x-2 \\ -1 & 1-x & 0 \\ 2-x & 0 & 0 \end{vmatrix}$$

$$\stackrel{\text{det}}{=} \underset{\text{Col 3}}{(x-2)} \begin{vmatrix} -1 & 1-x \\ 2-x & 0 \end{vmatrix} = (x-2) \left(- (1-x)(2-x) \right) \\
 = (2-x)^2 (1-x)$$

Calcul de $E_2 = \left\{ \vec{u} \begin{pmatrix} 2 \\ 3 \end{pmatrix}, / f(\vec{u}) = 2\vec{u} \right\}$:

$$f(\vec{u}) = 2\vec{u}$$

① \vec{u} est un vect propre de f signifie $f(\vec{u}) = a\vec{u}$

$$f(\vec{u}) = A(\vec{u}) = \dots = 4\vec{u}.$$

Donc \vec{u} est un vecteur propre pour la valeur propre 4.

③ $\begin{cases} a_{n+1} = 2a_n - b_n - c_n \\ b_{n+1} = -a_n + 2b_n - c_n \\ c_{n+1} = a_n + b_n + 4c_n \end{cases} \Leftrightarrow \underbrace{\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix}}_{U_{n+1}} = \underbrace{\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix}}_A \underbrace{\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}}_{U_n}$

$U_{n+1} = A U_n$. La suite $(U_n)_{n \geq 0}$ est géométrique, donc $U_n = A^n U_0 = A^n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

J'appelle $B = (\vec{i}, \vec{j}, \vec{k})$ la base canonique de \mathbb{R}^3

_____ f l'endomorphisme de \mathbb{R}^3 tel que $A = \text{Mat}_B(f)$

$$P(x) = \begin{vmatrix} 2-x & -1 & -1 \\ -1 & 2-x & -1 \\ 1 & 1 & 4-x \end{vmatrix}$$

$$P(x) = \begin{vmatrix} 2-x & -1 & -1 \\ -1 & 2-x & -1 \\ 1 & 1 & 4-x \end{vmatrix} = \begin{vmatrix} 3-x & -1 & -1 \\ x-3 & 2-x & -1 \\ 0 & 1 & 4-x \end{vmatrix}$$

$$= \begin{matrix} L_2+L_1 \\ \\ \end{matrix} \begin{vmatrix} 3-x & -1 & -1 \\ 0 & 1-x & -2 \\ 0 & 1 & 4-x \end{vmatrix} \begin{matrix} C_1-C_2 \\ \text{div} \\ C_1 \end{matrix} = (3-x) \begin{vmatrix} 1-x & -2 \\ 1 & 4-x \end{vmatrix}$$

$$\begin{aligned} &= (3-x) [(1-x)(4-x) + 2] = (3-x) [4 - 4x - x + x^2 + 2] \\ &= (3-x) [6 - 5x + x^2] \\ &\quad (2-x)(3-x) \\ &= (2-x)(3-x)^2 \end{aligned}$$

2 valeurs propres 2 et 3

Calcul de $E_3 = \{ \vec{u} \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix} \right), / f(\vec{u}) = 3\vec{u} \}$: $f(\vec{u}) = 3\vec{u} \Leftrightarrow A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 2x - y - z = 3x \\ -x + 2y - z = 3y \\ x + y + 4z = 3z \end{cases} \Leftrightarrow \begin{cases} -x - y - z = 0 \\ -x - y - z = 2x \Leftrightarrow x + y + z = 0 \\ x + y + z = 0 \end{cases} \text{ dim } 2.$$

$$\dim(E_3) = \text{mult}(3)$$

↑

La dimension de l'espace propre relatif à la valeur propre 3 est égale à la multiplicité de 3.

donc A est diagonalisable.

on choisit

$$\vec{u} = (1; -1; 0) \text{ et } \vec{v} = (-1; 0; 1)$$

$$f(\vec{u}) = 3\vec{u}$$

$$f(\vec{v}) = 3\vec{v}$$

Calcul de $E_2 = \{ \vec{w} \in \mathbb{R}^3 / f(\vec{w}) = 2\vec{w} \}$

$$f(\vec{w}) = 2\vec{w} \Rightarrow A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \Leftrightarrow \begin{cases} 2x - y - z = 2x \\ -x + 2y - z = 2y \\ x + y + 4z = 2z \end{cases} \Leftrightarrow \begin{cases} -y - z = 0 \\ -x - z = 0 \\ x + y + 2z = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -y - z = 0 \\ -x - z = 0 \\ x + y + 2z = 0 \end{cases} \xrightarrow{L_2 + L_3} \begin{cases} -y - z = 0 \\ -x - z = 0 \\ y + z = 0 \end{cases} \Leftrightarrow \begin{cases} x + z = 0 \\ y + z = 0 \\ z = 0 \end{cases} \text{ d'où } z = 0$$

Si $x = t$, alors $z = -t$ et $y = t$ $\vec{w} = (1; 1; -1)$ convient

$$\boxed{f(\vec{w}) = 2\vec{w}}$$

$$\mathcal{B}' = (\vec{u}, \vec{v}, \vec{w}) \quad A' = \text{Mat}_{\mathcal{B}'}(f) = \begin{pmatrix} f(\vec{u})_{\mathcal{B}'} & f(\vec{v})_{\mathcal{B}'} & f(\vec{w})_{\mathcal{B}'} \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$$3\vec{u} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}'}, \quad 2\vec{w} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}_{\mathcal{B}'}$$

$$\boxed{A' = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$

$$A'^n = \begin{pmatrix} 3^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}$$

où \mathcal{L} est la matrice de passage de \mathcal{B} à \mathcal{B}'

$$A^n = \mathcal{L} A'^n \mathcal{L}^{-1}$$

$$P = \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \\ B & B & B \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$
 On calcule P^{-1}
 Dans P , on a $\vec{u}, \vec{v}, \vec{w}$ en fonction de $\vec{i}, \vec{j}, \vec{k}$
 Dans P^{-1} , on aura $\vec{i}, \vec{j}, \vec{k}$ en fonction de $\vec{u}, \vec{v}, \vec{w}$

$$\rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$
 Col 3: $\vec{j} = \vec{v} + \vec{w} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}_B$. On vérifie OK
 Col 2: $\vec{i} - \vec{j} = \vec{u} \Rightarrow \vec{i} = \vec{u} + \vec{v} + \vec{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}_B$. Vérif. OK

Col 1: $-\vec{i} + \vec{k} = \vec{v} \Rightarrow \vec{k} = \vec{i} + \vec{v} = \vec{u} + 2\vec{v} + \vec{w} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}_B$

$$P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$P A^{-1} P^{-1} = P \begin{pmatrix} 3^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} P^{-1} = P \begin{pmatrix} 3^n & 0 & 3^n \\ 3^n & 3^n & 2 \cdot 3^n \\ 2^n & 2^n & 2^n \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^n & 2^n - 3^n & 2^n - 3^n \\ 2^n - 3^n & 2^n & 2^n - 3^n \\ 3^n - 2^n & 3^n - 2^n & 2 \cdot 3^n - 2^n \end{pmatrix} = A^{-1}$$

rappel

$$\begin{pmatrix} a_m \\ b_m \\ c_m \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \begin{pmatrix} 2^n & 2^n - 3^n & 2^n - 3^n \\ 2^n \cdot 3^n & 2^n & 2^n \cdot 3^n \\ 3 \cdot 2^n & 3 \cdot 2^n & 2 \cdot 3^n \cdot 2^n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 2^n - 2 \cdot 3^n \\ 3 \cdot 2^n - 2 \cdot 3^n \\ 4 \cdot 3^n - 3 \cdot 2^n \end{pmatrix}$$

$\xrightarrow{a_n}$
 $\xrightarrow{b_n}$
 $\xrightarrow{c_n}$

④ a

$$\begin{vmatrix} -x & -1 & -2 \\ -1 & 1-x & -1 \\ 2 & 1 & 4-x \end{vmatrix} = \begin{vmatrix} 1-x & 1-x & 1-x \\ -1 & 1-x & -1 \\ 2 & 1 & 4-x \end{vmatrix} \quad \begin{matrix} l_1 + l_2 + l_3 \\ \\ \\ \end{matrix}$$

$$= \begin{vmatrix} 1-x & 0 & 0 \\ -1 & 2-x & 0 \\ 2 & -1 & 2-x \end{vmatrix} = (1-x)(2-x)^2$$

$\begin{matrix} C_2 - C_1 & C_3 - C_1 \end{matrix}$

2 valeurs propres 1 et 2

Si $\dim(E_2) = 2$, alors f est diagonalisable } On calcule
 Si $\dim(E_2) = 1$, alors f n'est pas diagonalisable } $E_2 = \{ \vec{u} \mid f(\vec{u}) = 2\vec{u} \}$

$$f(\vec{u}) = \lambda \vec{u} \Rightarrow A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0 & -1 & -2 \\ -1 & 1 & -1 \\ 2 & 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -y - 2z = 2x \\ -x + y - z = 2y \\ 2x + y + 4z = 2z \end{cases} \Leftrightarrow \begin{cases} -2x - y - 2z = 0 \\ -x - y - z = 0 \\ 2x + y + 2z = 0 \end{cases} \Leftrightarrow \begin{cases} -2x - y - 2z = 0 \\ -x - y - z = 0 \\ x + z = 0 \end{cases}$$

dim(E_2) = 1 \neq mult(2), donc A n'est pas diagonalisable

$$\vec{u} = (1, 0, -1) \in E_2$$

On cherche \vec{v} tel que $f(\vec{v}) = 2\vec{v} + \vec{u} \Leftrightarrow A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$\begin{cases} -y - 2z = 2x + 1 \\ -x + y - z = 2y \\ 2x + y + 4z = 2z - 1 \end{cases} \Leftrightarrow \begin{cases} -2x - y - 2z = 1 \\ -x - y - z = 0 \\ 2x + y + 2z = -1 \end{cases} \Leftrightarrow \begin{cases} -x - y - z = 0 \\ x + z = -1 \end{cases}$$

Si $x = t$, alors $z = -1 - t$ et $y = -x - z = -t + 1 + t = 1$

$$x = 0, \quad z = -1, \quad y = 1$$

$$\vec{v} = (0; 1; -1)$$