

$$(9) S_n = \frac{1}{3} + \frac{1}{15} + \dots + \frac{1}{4n^2-1}$$

$$S_1 = \frac{1}{3} \quad S_2 = \frac{1}{3} + \frac{1}{15} = \frac{5+1}{15} = \frac{6}{15} = \frac{2}{5}$$

$$S_3 = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} = \frac{2}{5} + \frac{1}{35} = \frac{2 \times 7}{5 \times 7} + \frac{1}{35} = \frac{15}{35} = \frac{3}{7}$$

$$\boxed{S_1 = \frac{1}{3}} \quad \boxed{S_2 = \frac{2}{5}} \quad \boxed{S_3 = \frac{3}{7}} \quad \text{Il semble que } \boxed{S_n = \frac{n}{2n+1}}$$

On le prouve par récurrence :
Initialisation : Ok pour $n=1, 2$ et 3 .

Hérédité : Je prends un $n \geq 1$. Je suppose que $S_n = \frac{1}{3} + \frac{1}{15} + \dots + \frac{1}{4n^2-1} = \frac{n}{2n+1}$.

(On doit montrer que $S_{n+1} = \frac{1}{3} + \frac{1}{15} + \dots + \frac{1}{4(n+1)^2-1} = \frac{n+1}{2n+3}$)

$$\text{On ajoute } \frac{1}{4(n+1)^2-1} = \frac{1}{2^2(n+1)^2-1} = \frac{1}{(2n+2)^2-1} = \frac{1}{(2n+3)(2n+1)}$$

identité remarquable.

$$\begin{aligned} S_{n+1} &= S_n + \frac{1}{(2n+3)(2n+1)} = \frac{n}{2n+1} + \frac{1}{(2n+3)(2n+1)} = \frac{n(2n+3)}{(2n+1)(2n+3)} + \frac{1}{(2n+3)(2n+1)} \\ &= \frac{n(2n+3) + 1}{(2n+1)(2n+3)} = \frac{2n^2 + 3n + 1}{(2n+1)(2n+3)} = \frac{(n+1)(2n+1)}{(2n+1)(2n+3)} = \frac{n+1}{2n+3} \end{aligned}$$

Yes!

Conclusion : D'après le principe de récurrence : $S_n = \frac{n}{2n+1}$

Ainsi,

$$\boxed{S_{1000} = \frac{1000}{2001}}$$