

Combinatoire Coucou

(11) a) $I_{n+2} = \int_0^{\pi/2} \sin^{n+2} x dx = \int_0^{\pi/2} \sin x \cdot \sin^n x dx$ E.I.E | $u = \sin x$ $u' = (\cos x)$
| $v' = \sin x$ $v = -\cos x$

$$= \left[\sin^n x \cos x \right]_0^{\pi/2} + \int_0^{\pi/2} (n \sin^{n-1} x) \cos^2 x dx$$

$$= 0 + (n) \int_0^{\pi/2} \sin^{n-1} x (1 - \sin^2 x) dx$$

$$I_{n+2} = (n) \left(\int_0^{\pi/2} \sin^{n-1} x dx - \int_0^{\pi/2} \sin^{n+1} x dx \right)$$

$$= (n) (I_n - I_{n+2}) \Rightarrow I_{n+2} = (n) I_n - (n) I_{n+2}$$

$$\Rightarrow (n+2) I_{n+2} = (n) I_n \Rightarrow \boxed{I_{n+2} = \frac{n}{n+2} I_n}$$

b) $I_0 = \int_0^{\pi/2} 1 dx = [x]_0^{\pi/2} = \frac{\pi}{2}$ $\boxed{I_0 = \frac{\pi}{2}}$

$$I_1 = \int_0^{\pi/2} \sin x dx = [-\cos x]_0^{\pi/2} = 0 - (-1) = 1$$
 $\boxed{I_1 = 1}$

c) $I_2 = \frac{1}{2} I_0$ $I_4 = \frac{3}{4} I_2 = \frac{1 \cdot 3}{2 \cdot 4} I_0$ $I_6 = \frac{5}{6} I_4 = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} I_0$

Il semble que $\boxed{I_{2n} = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} \frac{\pi}{2}}$

$$I_3 = \frac{2}{3} I_1 = \frac{2}{3}$$
 $I_5 = \frac{4}{5} I_3 = \frac{2 \cdot 4}{3 \cdot 5}$ $\boxed{I_{2n+1} = \frac{2 \cdot 4 \dots (2n)}{3 \cdot 5 \dots (2n+1)}}$

En principe, on prouve ces 2 résultats par récurrence. C'est facile.
 Ensuite, on peut transformer les produits de pairs et les impairs comme suit:

$$2 \cdot 4 \dots (2n) = 2^n \cdot n! \quad \text{et} \quad 3 \cdot 5 \dots (2n-1) = \frac{(2n)!}{2^n \cdot n!}$$

Ainsi $I_{2n} = \frac{(2n)!}{2^{2n} (n!)^2}$ et $I_{2n+1} = \frac{2^{2n} (n!)^2}{(2n+1)!}$

(12) $u_0 = 2$ $u_1 = 7$ $u_{n+2} = 7u_{n+1} - 12u_n$ ($u_2 = 7u_1 - 12u_0 = 49 - 24 = 25$)

Initialisation: $u_0 = 4^0 + 3^0 = 2$ OK et $u_1 = 4^1 + 3^1 = 7$ OK

Hérédité: On prend un $n \geq 1$. On suppose que $u_n = 4^n + 3^n$ et $u_{n+1} = 4^{n+1} + 3^{n+1}$
 On montre que $u_{n+2} = 4^{n+2} + 3^{n+2}$

$$u_{n+2} = 7u_{n+1} - 12u_n = 7 \cdot 4^{n+1} + 7 \cdot 3^{n+1} - 12 \cdot 4^n - 12 \cdot 3^n$$

$$= 7 \cdot 4 \cdot 4^n + 7 \cdot 3 \cdot 3^n - 12 \cdot 4^n - 12 \cdot 3^n$$

$$= 7 \cdot 4^{n+1} + 7 \cdot 3^{n+1} - 4 \cdot 4^n - 4 \cdot 3^n$$

$$= 3 \cdot 4^{n+1} + 3 \cdot 3^{n+1} = 4 + 3^{n+2}$$
 OK

Conclusion: ...