

examen

①  $P(x) = 2x^4 + 32i$

$P(x) = 0 \Leftrightarrow 2x^4 = -32i = x^4 = -16i$



$x^4 = \underline{16e^{-i\pi/2}}$

les 4 racines sont les  $z_k = 16^{1/4} e^{i(\frac{\pi}{8} + \frac{2k\pi}{4})}$  où  $k=0,1,2,3$

$z_k = 2 e^{i(\frac{\pi}{8} + \frac{4k\pi}{8})}$

$z_0 = 2e^{-i\pi/8}$

$z_1 = 2e^{i3\pi/8}$

$z_2 = 2e^{i7\pi/8}$

$z_3 = 2e^{i11\pi/8} = 2e^{-i5\pi/8}$

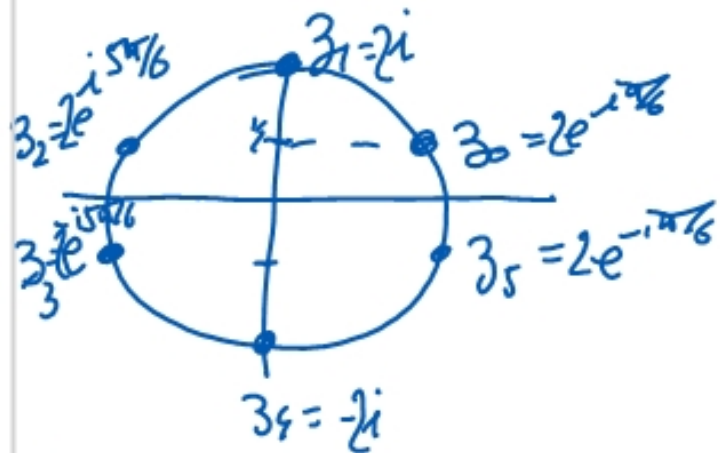
$2x^4 + 32i = 2(x - 2e^{-i\pi/8})(x - 2e^{i3\pi/8})(x - 2e^{i7\pi/8})(x - 2e^{i11\pi/8})$



ou  $(x - 2e^{-i5\pi/8})$

$$(2) X^6 + 64 = 0 \Leftrightarrow X^6 = -64 = 64e^{i\pi}$$

$$6 \text{ racis } z_k = 64^{1/6} e^{-i(\frac{\pi}{6} + \frac{2k\pi}{6})} = 2 e^{-i(\frac{\pi}{6} + \frac{2k\pi}{6})} \text{ cu } k=0,1,\dots,5$$



$$z_1 = 2e^{-i\pi/2}$$

$$X^6 + 64 = (X - 2e^{i\pi/6})(X - 2e^{-i\pi/6})(X - 2i)(X + 2i)(X - 2e^{i5\pi/6})(X - 2e^{-i5\pi/6})$$

$$= \left( X^2 - 2 \left( e^{i\pi/6} + e^{-i\pi/6} \right) X + 4 \right) (X^2 + 4) \left( X^2 - 2 \left( e^{i5\pi/6} + e^{-i5\pi/6} \right) X + 4 \right)$$

$2\cos(\frac{\pi}{6}) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$ 
 $2\cos(\frac{5\pi}{6}) = -\sqrt{3}$

$$X^6 + 64 = (X^2 - 2\sqrt{3}X + 4)(X^2 + 4)(X^2 + 2\sqrt{3}X + 4)$$

$$\textcircled{7} \left\{ \begin{array}{l} P(x) = (x-2)^{2n} + (x-1)^m - 1 = Q(x)(x^2 - 3x + 2) + R(x) \\ \deg(R) < 2 \Rightarrow R(x) = ax + b \end{array} \right.$$

$$x^2 - 3x + 2 = (x-1)(x-2)$$

valeur en 1:  $(1-2)^{2n} + 0 - 1 = Q(1) \cdot 0 + a + b \Rightarrow 0 = a + b$

valeur en 2:  $0 + (2-1)^m - 1 = Q(2) \cdot 0 + 2a + b \Rightarrow 0 = 2a + b$

$$\left\{ \begin{array}{l} a + b = 0 \\ 2a + b = 0 \end{array} \right. \Rightarrow a = b = 0, \text{ donc } \underline{R(x) = 0}$$

la division tombe juste

donc le poly  $Q(x)$  est divisible par  $x^2 - 3x + 2$

$$\textcircled{8} \left\{ \begin{array}{l} X^{m+1} - X^m - X + 1 = Q(x)(X-1)^2 + R(x) \\ \deg(R) < 2 \Rightarrow R(x) = ax + b \end{array} \right.$$

valeur en 1:  $1 \cdot 1 - 1 + 1 = Q(1) \cdot 0 + a + b \Rightarrow \underline{a + b = 0}$

On dérive:

$$(m+1)X^m - mX^{m-1} - 1 = Q'(x) \cdot (X-1)^2 + Q(x) \cdot 2(X-1) + \overset{R'(x)}{\downarrow} a$$

valeur en 1:  $m+1 - m - 1 = 0 + 0 + a \quad \underline{a = 0}$

$$\begin{cases} a + b = 0 \\ a = 0 \end{cases} \Rightarrow a = b = 0 \quad \text{le reste est nul, donc}$$

la division tombe juste.

Donc  $X^{m+1} - X^m - X + 1$  est divisible par  $(X-1)^2$

17 • }  $P(x) = (x-3)^{2n} - 2(x-2)^n + 1 = Q(x) \cdot \underline{(x-2)} + R(x)$   
 division par  $x-2$   
 $\deg(R) < \deg(x-2) = 1 \Rightarrow \underline{R(x) = a}$

valeur en 2:  $(-1)^{2n} - 0 + 1 = 0 + a \Rightarrow a = 2$

$$\boxed{R(x) = 2}$$

division par  $(x-2)(x-3)$ :

$P(x) = (x-3)^{2n} - 2(x-2)^n + 1 = Q(x) \cdot \underline{(x-2)(x-3)} + R(x)$   
 $\deg(R) < 2 \Rightarrow \underline{R(x) = ax + b}$

valeur en 2:  $(-1)^{2n} + 1 = 2a + b$

$$\begin{cases} 2a + b = 2 \\ 3a + b = -1 \end{cases} \Rightarrow \begin{cases} 2a + b = 2 \\ a = -3 \frac{1}{2} - b \end{cases}$$

valeur en 3:  $0 - 2 \cdot 1^n + 1 = 3a + b$

$$\Rightarrow b = 8$$

$$\boxed{R(x) = -3x + 8}$$

division par  $(x-2)^2$  :  $\overbrace{R(x)}^{\text{de degré } < 2}$

$$(x-3)^{2n} - 2(x-2)^{2n} + 1 = Q(x)(x-2)^2 + ax + b.$$

valeurs en 2 :  $(-1)^{2n} + 1 = 0 + 2a + b$        $2a + b = 2$

On dérive :

$$2n(x-3)^{2n-1} - 2n(x-2)^{2n-1} = Q'(x)(x-2)^2 + Q(x)2(x-2) + a$$

valeurs en 2 :

$$2n(-1)^{2n-1} - 0 = 0 + 0 + a$$

$$\text{Or } (-1)^{2n-1} = -1 \Rightarrow a = -2n$$

$$\Rightarrow b = 2 - 2a = 2 + 4n$$

$$R(x) = -2nX + 2 + 4n$$

$$\textcircled{1} \quad \mathcal{L}(x) = 2x^4 + 32i$$

$$\mathcal{L}(x) = 0 \Leftrightarrow 2x^4 = -32i \Leftrightarrow x^4 = -16i$$



$$x^4 = 16e^{-i\pi/2}$$

$\textcircled{\text{II}}$  Les racines  $n^{\text{ième}}$  de  $Re^{i\theta}$  (ici  $16e^{-i\pi/2}$ ) sont les  $z_k = R^{1/n} e^{i(\frac{\theta}{n} + 2k\pi/n)}$

$$\text{ici } z_k = 16^{1/4} e^{-i\frac{\pi}{8} + i\frac{2k\pi}{4}} \quad \text{où } k = 0, 1, 2, 3$$

$$\text{or } 16^{1/4} = (2^4)^{1/4} = 2$$

$$z_0 = 2e^{-i\pi/8}$$

$$z_1 = 2e^{i3\pi/8}$$

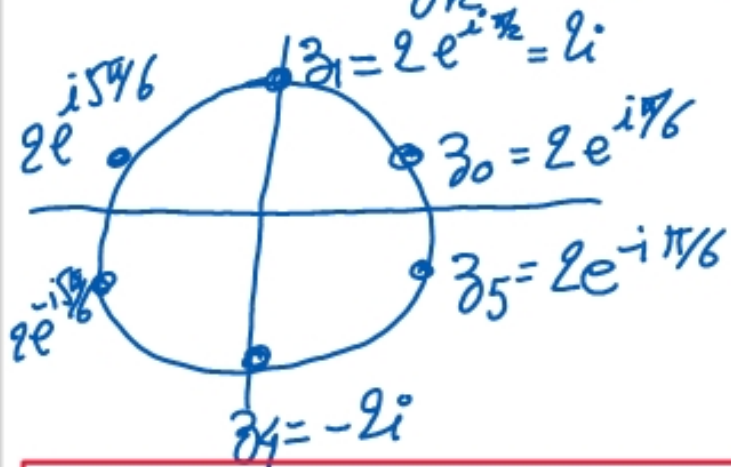
$$z_2 = 2e^{i7\pi/8}$$

$$z_3 = 2e^{i11\pi/8}$$

$$2x^4 + 32i = 2(x - 2e^{-i\pi/8})(x - 2e^{i3\pi/8})(x - 2e^{i7\pi/8})(x - 2e^{i11\pi/8})$$

$$\text{ou } e^{-i\frac{5\pi}{8}}$$

②  $X^6 + 64 = 0 \Leftrightarrow X^6 = -64 = 64 e^{i\pi}$  (les 6 racines sont les  
 $z_k = 64^{1/6} e^{i(\frac{\pi}{6} + 2k\frac{\pi}{6})} = 2 e^{i(\frac{\pi}{6} + 2k\frac{\pi}{6})}$  où  $k = 0, 1, \dots, 5$ )



$$X^6 + 64 = (X - 2e^{i\pi/6})(X - 2e^{-i\pi/6})(X - 2i)(X + 2i)(X - 2e^{i5\pi/6})(X - 2e^{-i5\pi/6})$$

$$= \left( X^2 - 2(e^{i\pi/6} + e^{-i\pi/6})X + 4 \right) (X^2 + 4) \left( X^2 - 2(e^{i5\pi/6} + e^{-i5\pi/6})X + 4 \right)$$

$2\cos(\frac{\pi}{6}) = \sqrt{3}$ 
 $2\cos(\frac{5\pi}{6}) = -\sqrt{3}$

$$X^6 + 64 = (X^2 - 2\sqrt{3}X + 4)(X^2 + 4)(X^2 + 2\sqrt{3}X + 4)$$

dans  $\mathbb{R}[X]$



$$\textcircled{7} \quad P(x) = (x-2)^{2m} + (x-1)^n - 1 = Q(x)(x^2 - 3x + 2) + R(x)$$

$$\left. \begin{array}{l} \text{deg}(R) < 2 \\ \Rightarrow R(x) = ax + b \end{array} \right\}$$

$$x^2 - 3x + 2 = (x-1)(x-2)$$

valeurs en 1:  $(1-2)^{2m} + 0 - 1 = Q(1) \cdot 0 + a + b \Rightarrow 0 = a + b$

valeurs en 2:  $0 + (2-1)^n - 1 = Q(2) \cdot 0 + 2a + b \Rightarrow 0 = 2a + b$

$$\begin{cases} a + b = 0 \\ 2a + b = 0 \end{cases} \Rightarrow a = b = 0, \text{ donc } \underline{R(x) = 0}$$

La division tombe juste.

donc le poly  $Q(x)$  est divisible par  $x^2 - 3x + 2$

$$\textcircled{8} \begin{cases} X^{n+1} - X^n - X + 1 = Q(x)(X-1)^2 + R(x) \\ \deg(R) < 2 \Rightarrow R(x) = aX + b \end{cases}$$

valeurs en 1:  $1 - 1 - 1 + 1 = Q(1) \cdot 0 + a + b \Rightarrow \underline{a + b = 0}$

On dérive:

$$(n+1)X^n - nX^{n-1} - 1 = Q'(x) \cdot (X-1)^2 + Q(x) \cdot 2(X-1) + \overset{R'(x)}{\downarrow} a$$

valeurs en 1:  $(n+1) - n - 1 = Q'(1) \cdot 0 + Q(1) \cdot 0 + a \Rightarrow \underline{a = 0}$

donc  $a = b = 0$   $R(x) = 0$

la division tombe juste

donc  $X^{n+1} - X^n - X + 1$  est divisible par  $(X-1)^2$

(17) division par  $X-2$ :

$$P(x) = (x-3)^{2m} - 2(x-2)^m + 1 = Q(x)(x-2) + R(x)$$

et  $\deg(R) < \deg(x-2) = 1 \Rightarrow R(x) = a$

valeur en 2:  $(-1)^{2m} - 0 + 1 = 0 + a \Rightarrow a = 2$

$$R(x) = 2$$

division par  $(x-2)(x-3)$

$$\begin{cases} P(x) = (x-3)^{2m} - 2(x-2)^m + 1 = Q(x)(x-2)(x-3) + R(x) \\ \deg(R) < 2 \Rightarrow R(x) = ax + b \end{cases}$$

valeur en 2:  $(-1)^{2m} + 1 = 2a + b$

valeur en 3:  $0 - 2 \cdot 1^m + 1 = 3a + b$

$$\begin{cases} 2a + b = 2 \\ 3a + b = -1 \end{cases} \Rightarrow \begin{cases} 2a + b = 2 \\ a = -3 \end{cases}$$

$$\Rightarrow b = 8$$

$$R(x) = -3x + 8$$

division par  $(x-2)^2$

$R(x)$  est de degré  $< 2$

$$(x-3)^{2n} - 2(x-2)^n + 1 = Q(x)(x-2)^2 + aX + b$$

valeur en 2:  $(-1)^{2n} + 1 = 0 + 2a + b$

$2a + b = 2$

On dérive:

$$2n(x-3)^{2n-1} - 2n(x-2)^{n-1} = Q'(x)(x-2)^2 + Q(x)2(x-2) + a$$

valeur en 2:

$$2n(-1)^{2n-1} - 0 = 0 + 0 + a$$

or  $(-1)^{2n-1} = -1 \Rightarrow a = -2n$

$\Rightarrow b = 2 - 2a = 2 + 4n$

$R(x) = -2nX + 2 + 4n$