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$$\frac{1}{k^2-1} = \frac{1}{(k+1)(k-1)} = \frac{a}{k+1} + \frac{b}{k-1}$$

$$a = \frac{1}{k+1} \Big|_{k=-1} = -\frac{1}{2} \quad b = \frac{1}{k-1} \Big|_{k=1} = \frac{1}{2} \quad \left(\text{voir cours et TD sur la réduction en dt simples} \right)$$

donc

$$\sum_{k=1}^n \frac{1}{k^2-1} = \sum_{k=1}^n \left(\frac{-1/2}{k+1} + \frac{1/2}{k-1} \right)$$

$$= -\frac{1}{2} \sum_{k=1}^n \frac{1}{k+1} + \frac{1}{2} \sum_{k=1}^n \frac{1}{k-1}$$

$$= -\frac{1}{2} \sum_{k=1}^n \frac{1}{k+1} + \frac{1}{2} \sum_{k=0}^{n-1} \frac{1}{k+1} \quad \leftarrow \text{décalage}$$

$$= -\frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{k+1} - \frac{1}{2} \frac{1}{n+1} + \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{k+1} + \frac{1}{2} \frac{1}{1}$$

$$= \frac{1}{2} - \frac{1}{2(n+1)} = \frac{2n-1}{2(n+1)} = \frac{n}{n+1}$$