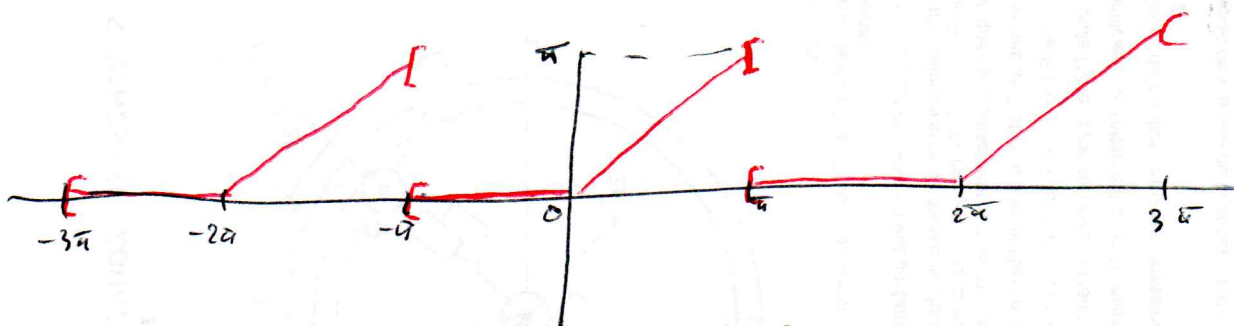


**Exo 2**  $f(x) = \begin{cases} 0 & \text{si } x \in [-\pi, 0] \\ x & \text{si } x \in [0, \pi] \end{cases}$  et  $f$  est  $2\pi$ -périodique



②  $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_0^{\pi} x dx = \frac{1}{2\pi} \left[ \frac{x^2}{2} \right]_0^{\pi} = \frac{\pi^2}{4\pi} = \frac{\pi}{4}$

• pour  $n \neq 0$   $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \int_0^{\pi} x \cos(nx) dx$

IPP |  $u = x$   $u' = 1$   
 $v = \cos(nx)$   $v' = -n \sin(nx)$

$$a_n = \frac{1}{\pi} \left[ x \frac{\sin(nx)}{n} \right]_0^{\pi} - \frac{1}{\pi} \int_0^{\pi} \frac{\sin(nx)}{n} dx = -\frac{1}{\pi} \left[ -\frac{\cos(nx)}{n^2} \right]_0^{\pi}$$

$$a_n = -\frac{1}{\pi} \left[ -\frac{\cos(n\pi)}{n^2} + \frac{\cos(0)}{n^2} \right] = -\frac{1}{\pi} \left[ -\frac{(-1)^n}{n^2} + \frac{1}{n^2} \right] = \frac{(-1)^n - 1}{n^2 \pi} = \begin{cases} \frac{2}{n^2 \pi} & \text{qd } n \text{ est pair} \\ 0 & \text{qd } n \text{ est pair} \end{cases}$$

•  $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_0^{\pi} x \sin(nx) dx$

IPP |  $u = x$   $u' = 1$   
 $v = \sin(nx)$   $v' = n \cos(nx)$

$$b_n = \frac{1}{\pi} \left[ -x \frac{\cos(nx)}{n} \right]_0^{\pi} + \frac{1}{\pi} \int_0^{\pi} \frac{\cos(nx)}{n} dx$$

$$= \frac{1}{\pi} \left[ -\frac{\pi \cos(n\pi)}{n} \right] + \frac{1}{\pi} \left[ \frac{\sin(nx)}{n^2} \right]_0^{\pi} = \frac{-(-1)^n \pi}{n} = \frac{(-1)^{n+1} \pi}{n} = b_n$$

• Duichlet. Si  $f$  est périodique et de classe  $C^1$  par morceaux (c'est le cas ici!), alors  $S_f(x)$  converge vers  $\frac{f(x^-) + f(x^+)}{2}$

Si de plus  $f$  est continue en  $x$ , alors  $S_f(x) \rightarrow f(x)$

③  $S_f(x) = a_0 + \sum_{n=1}^{+\infty} (a_n \cos(nx) + b_n \sin(nx))$

En  $x=0$ ,  $f$  est continue, donc  $S_f(0) = f(0) = 0$

$$S_f(0) = \frac{\pi}{4} + \sum_{n=1}^{+\infty} a_n = \frac{\pi}{4} + \sum_{p=0}^{+\infty} \frac{-2}{(2p+1)^2 \pi} = 0 \Rightarrow \frac{2}{\pi} \sum_{p=0}^{+\infty} \frac{1}{(2p+1)^2} = \frac{\pi}{4}$$

$$\Rightarrow \sum_{p=0}^{+\infty} \frac{1}{(2p+1)^2} = \frac{\pi^2}{8}$$

$$\begin{aligned}
 \textcircled{4} \quad S &= \sum_{n=1}^{+\infty} \frac{1}{n^2} = \sum_{n \text{ pair}} \frac{1}{n^2} + \sum_{n \text{ impair}} \frac{1}{n^2} \\
 &= \sum_{p=1}^{+\infty} \frac{1}{(2p)^2} + \frac{\pi^2}{8} \\
 &= \frac{1}{4} \sum_{p=1}^{+\infty} \frac{1}{p^2} + \frac{\pi^2}{8} = \frac{1}{4} S + \frac{\pi^2}{8}
 \end{aligned}$$

$$S = \frac{1}{4} S + \frac{\pi^2}{8} \Rightarrow S - \frac{1}{4} S = \frac{\pi^2}{8} \Rightarrow S \left(1 - \frac{1}{4}\right) = \frac{\pi^2}{8} \Rightarrow S \frac{3}{4} = \frac{\pi^2}{8} \Rightarrow S = \frac{\pi^2}{8} \cdot \frac{4}{3} = \frac{\pi^2}{6}$$

⑤. Si  $f$  est périodique et continue par morceaux (c'est le cas ici !)

$$\text{alors} \quad \frac{1}{T} \int_{[0, T]} f(x) dx = a_0 + \frac{1}{2} \sum_{n=1}^{+\infty} (a_n^2 + b_n^2)$$

$$\frac{1}{2\pi} \int_0^{2\pi} x^2 dx = \left(\frac{\pi}{4}\right)^2 + \frac{1}{2} \left( \sum_{p=1}^{+\infty} \frac{4}{(2p+1)^2} + \sum_{n=1}^{+\infty} \frac{1}{n^2} \right)$$

$$\frac{1}{2\pi} \left[ \frac{x^3}{3} \right]_0^{2\pi} = \frac{\pi^2}{16} + \frac{2}{\pi^2} \sum_{p=1}^{+\infty} \frac{1}{(2p+1)^2} + \frac{1}{2} \frac{\pi^2}{6}$$

$$\frac{1}{2\pi} \frac{\pi^3}{3} = \frac{3\pi^2}{3 \times 16} + \frac{4\pi^2}{3 \times 16} + \frac{2}{\pi^2} \sum_{p=1}^{+\infty} \frac{1}{(2p+1)^2}$$

$$\frac{8\pi^2}{3 \times 16} - \frac{7\pi^2}{3 \times 16} = \frac{2}{\pi^2} \sum_{p=1}^{+\infty} \frac{1}{(2p+1)^2} \Rightarrow \frac{\pi^2}{48} = \frac{2}{\pi^2} \sum_{p=1}^{+\infty} \frac{1}{(2p+1)^2}$$

$$\Rightarrow \boxed{\frac{\pi^4}{96} = \sum_{p=1}^{+\infty} \frac{1}{(2p+1)^4}}$$