

Problème 1 :

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

① $A^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 8 & 0 & 5 \end{pmatrix}$ $25+16 = \epsilon_1$

$A^3 = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 8 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 41 & 0 & 20 \\ 0 & 1 & 0 \\ 20 & 0 & 41 \end{pmatrix}$

② Dans \mathcal{B} , $f(u) = K \begin{pmatrix} \sqrt{a} \\ 0 \\ -\sqrt{b} \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{a} \\ 0 \\ -\sqrt{b} \end{pmatrix} = \begin{pmatrix} -a\sqrt{b} \\ 0 \\ \sqrt{a}b \end{pmatrix}$ donc $\boxed{f(u) = -\sqrt{ab} \vec{u}}$
 et $-\sqrt{ab} \vec{u} = -\sqrt{ab} \begin{pmatrix} \sqrt{a} \\ 0 \\ -\sqrt{b} \end{pmatrix} = \begin{pmatrix} -a\sqrt{b} \\ 0 \\ \sqrt{a}b \end{pmatrix}$

③ $K \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ c.a. $\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ c.a. $\begin{cases} az=0 \\ 0=0 \\ bx=0 \end{cases} \Rightarrow \begin{cases} z=0 \\ x=0 \end{cases}$ car a et $b \neq 0$
 $\boxed{\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$ $\boxed{\vec{v} \in \ker(f)}$

③ $f(\vec{w}) = \sqrt{ab} \vec{w}$ c.a. $\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sqrt{ab} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ c.a. $\begin{cases} az = \sqrt{ab} x \\ 0 = \sqrt{ab} y \\ bx = \sqrt{ab} z \end{cases} \Leftrightarrow \begin{cases} -\sqrt{ab} x + az = 0 \\ y = 0 \\ bx - \sqrt{ab} z = 0 \end{cases}$
 L_1 c.a. $z = \frac{\sqrt{b}}{a} x$ et L_2 c.a. $z = \frac{b}{\sqrt{ab}} x = \frac{\sqrt{b}}{a} x$ Si $x = \sqrt{a}$: $z = \sqrt{b}$

$$\boxed{\vec{w} = (\sqrt{a}; 0; \sqrt{b})}$$

④ $\begin{pmatrix} \sqrt{a} & 0 & \sqrt{a} \\ 0 & 1 & 0 \\ -\sqrt{b} & 0 & \sqrt{b} \end{pmatrix} \rightarrow \begin{pmatrix} \sqrt{a} & 0 & 0 \\ 0 & 1 & 0 \\ -\sqrt{b} & 0 & 2\sqrt{b} \end{pmatrix}$ le pivot est terminé. Il n'y a pas de colonnes de 0, donc les vecteurs sont liés.
 \mathcal{B}' est une base de \mathbb{R}^3

⑤ $\mathcal{P} = \begin{pmatrix} \text{coord de } \vec{u} \\ \text{dans } \mathcal{B} & \text{coord de } \vec{v} \\ \text{dans } \mathcal{B} & \text{coord de } \vec{w} \\ \text{dans } \mathcal{B} & \end{pmatrix} = \begin{pmatrix} \sqrt{a} & 0 & \sqrt{a} \\ 0 & 1 & 0 \\ -\sqrt{b} & 0 & \sqrt{b} \end{pmatrix} = \mathcal{P}$

$$\textcircled{6} \text{Mat}_{\mathcal{B}}(f) = \begin{pmatrix} f(\vec{u}) & f(\vec{v}) & f(\vec{w}) \\ \mathcal{B} & \mathcal{B} & \mathcal{B} \end{pmatrix}$$

$$\text{or } f(\vec{u}) = -\sqrt{ab} \vec{u} = \begin{pmatrix} -\sqrt{ab} \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}}, \quad f(\vec{v}) = \vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}}$$

$$\text{et } f(\vec{w}) = \sqrt{ab} \vec{w} = \begin{pmatrix} 0 \\ \sqrt{ab} \\ 0 \end{pmatrix}_{\mathcal{B}} \quad \text{Donc } \boxed{\text{Mat}_{\mathcal{B}}(f) = \begin{pmatrix} -\sqrt{ab} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{ab} \end{pmatrix}}$$

$$\textcircled{7} f(\vec{a}) = d\vec{a} \text{ et } \forall \alpha \in \mathbb{R}, f(\alpha \vec{a}) = \alpha \vec{a}$$

$$\text{Donc } (\alpha f + \beta \text{Id}_{\mathbb{R}^3})(\vec{a}) = \alpha f(\vec{a}) + \beta \vec{a} = \alpha d \vec{a} + \beta \vec{a} = (\alpha d + \beta) \vec{a}$$

$$\textcircled{8} A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = k + I$$

$$\text{avec } \boxed{\alpha = \beta = 1} \text{ et } \boxed{a = 4} \text{ et } \boxed{b = 4}$$

On appelle g l'endomorphisme tel que $A = \text{Mat}_{\mathcal{B}}(g)$

$$g = f + \text{Id}_{\mathbb{R}^3}$$

$$f(\vec{u}) = -\sqrt{ab} \vec{u} \text{ c'ad } f(\vec{u}) = -2\vec{u} \quad \text{donc } g(\vec{u}) = -2\vec{u} + \vec{u} = -\vec{u}$$

$$f(\vec{v}) = \vec{0} \quad \text{donc } g(\vec{v}) = \vec{0} + \vec{v} = \vec{v}$$

$$f(\vec{w}) = \sqrt{ab} \vec{w} = 2\vec{w} \quad \text{donc } g(\vec{w}) = 2\vec{w} + \vec{w} = 3\vec{w}$$

$$\text{Donc } \boxed{\text{Mat}_{\mathcal{B}}(g) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}} \text{ donc } \underline{A \text{ est diagonalisable}}$$

$$\textcircled{9} Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 2 \end{pmatrix} \begin{array}{l} \text{dans } \mathcal{Q}, \text{ on a } \vec{u}, \vec{v}, \vec{w} \text{ en fonction de } \vec{i}, \vec{j}, \vec{k} \\ \text{dans } \mathcal{Q}', \text{ on a } \vec{i}, \vec{j}, \vec{k} \end{array} \quad \vec{u}, \vec{v}, \vec{w}$$

$$\text{colonne 3: } 4\vec{k} = \vec{w} - \vec{u} \quad \left[\vec{k} = -\frac{1}{4} \vec{u} + \frac{1}{4} \vec{w} \right] \text{ on vérifie OK}$$

$$Q \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 4 \end{pmatrix} \begin{array}{l} \vec{w} - \vec{u} \\ \vec{v} \end{array} \quad \text{colonne 2: } \boxed{\vec{j} = \vec{v}}$$

$$\text{colonne 1: } \vec{v} - 2\vec{k} = \vec{u} \Rightarrow \vec{u} = \vec{v} - 2\vec{k} \Rightarrow \left[\vec{u} = \frac{1}{2} \vec{v} + \frac{1}{2} \vec{w} \right] \text{ on vérifie OK}$$

$$\boxed{Q^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix}}$$

10) Q est la matrice de passage de B à B'

$$\Delta = Q^{-1} A Q = \text{Mat}_{B'}(A) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{le calcul de } Q^{-1} A Q \text{ est inutile !}$$

$$\Delta^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^n \end{pmatrix} \quad \text{car } \Delta \text{ est diagonal}$$

11) pour $n=1$: $\Delta = Q^{-1} A Q \Rightarrow Q \Delta = Q Q^{-1} A Q$
 $= I_3 A Q = A Q$
 $\Rightarrow Q \Delta Q^{-1} = A Q Q^{-1} = A I_3 = A \quad \text{OK}$

Remarque: Je prends un $n \geq 1$. Je suppose que $A^n = Q \Delta^n Q^{-1}$

$$A^{n+1} = A \cdot A = (Q \Delta^n Q^{-1})(Q \Delta Q^{-1}) = Q \Delta^n (Q^{-1} Q) \Delta Q^{-1} = Q \Delta^n I_3 \Delta Q^{-1} = Q \Delta^{n+1} Q^{-1}$$

$$\Rightarrow A^{n+1} = Q \Delta^{n+1} Q^{-1} \quad \text{CQFD}$$

Conclusion: D'après le principe de récurrence, $\forall n \in \mathbb{N}$, $A^n = Q \Delta^n Q^{-1}$

$$A^n = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(-1)^n & 0 & -\frac{1}{2}(-1)^n \\ 0 & 1 & 0 \\ \frac{1}{2}3^n & 0 & \frac{1}{2}3^n \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2}(-1)^n & 0 & -\frac{1}{2}(-1)^n \\ 0 & 1 & 0 \\ \frac{1}{2}3^n & 0 & \frac{1}{2}3^n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-11^n + 3^n}{2} & 0 & \frac{3^n - (-11^n)}{2} \\ 0 & 1 & 0 \\ 3^n - (-11^n) & 0 & \frac{3^n + (-11^n)}{2} \end{pmatrix} = A^n$$

Problem 1 (21) $v_0 = u_1 - \frac{1}{2}u_0 = \frac{1}{2} - \frac{1}{2}(-1) = \frac{1}{2} + \frac{1}{2} = 1$ $\boxed{v_0 = 1}$

(2) $u_2 = u_1 - \frac{1}{4}u_0 = \frac{1}{2} - \frac{1}{4}(-1) = \frac{1}{2} + \frac{1}{4} = \frac{2+1}{4} = \frac{3}{4}$ $\boxed{u_2 = \frac{3}{4}}$

$u_1 - u_0 = \frac{1}{2} + 1 = \frac{3}{2}$ $u_2 - u_1 = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ $u_1 - u_0 \neq u_2 - u_1$, donc

La suite $(u_n)_{n \geq 0}$ n'est pas arithmétique

$\frac{u_1}{u_0} = \frac{\frac{1}{2}}{-1} = -\frac{1}{2}$ et $\frac{u_2}{u_1} = \frac{\frac{3}{4}}{\frac{1}{2}} = \frac{3}{2}$

$\frac{u_2}{u_1} \neq \frac{u_1}{u_0}$, donc la suite $(u_n)_{n \geq 0}$ n'est pas géométrique

(22) $v_{n+1} = u_{n+2} - \frac{1}{2}u_{n+1} = u_{n+1} - \frac{1}{2}u_n - \frac{1}{2}u_{n+1} = \frac{1}{2}u_{n+1} - \frac{1}{2}u_n = \frac{1}{2}(u_{n+1} - u_n)$

$\boxed{v_{n+1} = \frac{1}{2}v_n}$ - Donc $(v_n)_{n \geq 0}$ est une suite géométrique

(23) $\sigma_n = \left(\frac{1}{2}\right)^n \sigma_0 = \left(\frac{1}{2}\right)^n$ $\boxed{\sigma_n = \left(\frac{1}{2}\right)^n}$

(31) $\omega_0 = \frac{u_0}{v_0} = \frac{-1}{1} = -1$ v_n ne s'annule jamais! Pas de problème pour cet.

(32) $\omega_{n+1} = \frac{u_{n+1}}{\sigma_{n+1}} = \frac{u_n + \frac{1}{2}u_n}{\frac{1}{2}\sigma_n} = 2 + \frac{u_n}{\sigma_n} = 2 + \omega_n$

Donc $(\omega_n)_{n \geq 0}$ est une suite arithmétique

(33) $\omega_n = \omega_0 + 2n$ $\boxed{\omega_n = 2n - 1}$

(4) $\omega_n = \frac{u_n}{\sigma_n} \Rightarrow u_n = \omega_n \cdot \sigma_n = (2n-1) \cdot \frac{1}{2^n}$ $\boxed{u_n = \frac{2n-1}{2^n}}$

(5) Pour $n=0$: $S_n = u_0 = -1$ et $2 - \frac{0+3}{2^0} = 2-3 = -1$ ok

Hérédité: On prend un $n \geq 0$. On suppose par $S_n = 2 - \frac{2n+3}{2^n}$

$S_{n+1} = u_0 + u_1 + \dots + u_{n+1} = S_n + u_{n+1} = S_n + \frac{2(n+1)-1}{2^{n+1}} = S_n + \frac{2n+1}{2^{n+1}}$

$= 2 - \frac{2n+3}{2^n} + \frac{2n+1}{2^{n+1}} = 2 - \frac{2(2n+3)}{2^{n+1}} + \frac{2n+1}{2^{n+1}} = 2 - \frac{4n+6-(2n+1)}{2^{n+1}} = 2 - \frac{2n+5}{2^{n+1}}$

$= 2 - \frac{2(n+1)+3}{2^{n+1}}$ CQFD

Conclusion: D'après le principe de récurrence:

$\forall n \geq 0, S_n = 2 - \frac{2n+3}{2^n}$

(51). $\Delta = 1 - 1 = 0$. Il y a une racine double.

$$x = \frac{1}{2}$$

(52)

$$u_n = (An + B) \frac{1}{2^n}$$

(53)

$$u_0 = B = -1$$

$$u_1 = \frac{1}{2} = \frac{A+B}{2} \Rightarrow A+B=1 \Rightarrow A = 1 - B = 2$$

$$u_n = \frac{(2n-1)}{2^n}$$