

les eqns diff :

① $2xy' - 3y = 4x^5$

Holt: $2xy' - 3y = 0$ $y_H = A e^{\int \frac{3}{2x} dx} = A e^{\frac{3}{2} \ln x} = A x^{\frac{3}{2}}$ $y_H = A x^{\frac{3}{2}}$ où $A \in \mathbb{R}$

Sol part: MVC $y_2 = A x^{\frac{3}{2}}$ $y_2' = A \frac{3}{2} x^{\frac{1}{2}} + \frac{3}{2} A x^{-\frac{1}{2}}$
 $2xy_2' - 3y_2 = 2A \frac{3}{2} x^{\frac{3}{2}+1} + 3A x^{\frac{3}{2}+1} - 3A x^{\frac{3}{2}} = 2A \frac{3}{2} x^{\frac{5}{2}} = 4x^5 \Rightarrow A \frac{3}{2} x^{\frac{5}{2}} = 4x^5 \Rightarrow A \frac{3}{2} x^{\frac{5}{2}} = 2x^{\frac{5}{2}}$

$\Rightarrow A x^{\frac{3}{2}} = \int 2x^{\frac{5}{2}} dx = 2 \int x^{\frac{5}{2}} dx = 2 \frac{x^{\frac{7}{2}}}{\frac{7}{2}} = \frac{4}{7} x^{\frac{7}{2}} \Rightarrow y_2 = \frac{4}{7} x^{\frac{7}{2}} \cdot x^{-\frac{3}{2}} = \frac{4}{7} x^2 = y_2$

Sols de l'eq complete: $y = A x^{\frac{3}{2}} + \frac{4}{7} x^2$ où $A \in \mathbb{R}$

On doit utiliser la MVC car le coeff devant y' n'est pas constant.

② $4y'' + 4y' + y = x^2 + 9x + 9$

Holt: $P(x) = 4x^2 + 4x + 1 = (2x+1)^2$ $-\frac{1}{2}$ est racine double

$y_H = (Ax+B)e^{-\frac{1}{2}x}$

Sol part: $y_2 = ax^2 + bx + c$ $y_2' = 2ax + b$ $y_2'' = 2a$

$4y_2'' + 4y_2' + y_2 = 8a + 8ax + 4b + ax^2 + bx + c = ax^2 + (b+8a)x + 8a+4b+c$
 $= x^2 + 9x + 9$

$\begin{cases} a=1 \\ b+8a=9 \\ 8a+4b+c=9 \end{cases} \Rightarrow \begin{cases} b=1 \\ c=9-8-4=-3 \end{cases}$

$y_2 = x^2 + x - 3$

Sols de l'eq complete: $y = (Ax+B)e^{-\frac{1}{2}x} + x^2 + x - 3$ où $A, B \in \mathbb{R}$

les complexes

① $P(z) = iz^2 - 5z + 1 - 7i$

$\Delta = (-5)^2 - 4i(1-7i) = 25 - 4i - 28 = -3 - 4i = (a+bi)^2 = a^2 - b^2 + i2ab$

$\begin{cases} a^2 - b^2 = -3 \\ 2ab = -4 \\ |a+bi| = \sqrt{a^2+b^2} = 5 \end{cases}$

$4_1 + 4_2: 2a^2 = 2 \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$

$4_3 - 4_1: 2b^2 = 8 \Rightarrow b^2 = 4 \Rightarrow b = \pm 2$

$4_2: a \text{ et } b \text{ sont de signe opposé} \quad \sqrt{\Delta} = \pm (1 - 2i)$

$z = \frac{+5 \pm (1-2i)}{2i} = \begin{cases} \frac{6-2i}{2i} = -1-3i \\ \frac{4+2i}{2i} = 1-2i \end{cases}$

② $\left(\frac{1-i}{i}\right)^4 = \left(\frac{1-i}{i}\right)^4 = (1-i)^4 = \left(\sqrt{2} \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right)^4 = \left(4 \left(e^{-i\pi/4}\right)\right)^4 = 4e^{-i\pi} = -4$

$\frac{2+3i}{5-i} = \frac{(2+3i)(5+i)}{(5-i)(5+i)} = \frac{10+2i+15i-3}{25+1} = \frac{7}{26} + \frac{17i}{26}$

③ $z_1 = 1 - i\sqrt{3} = 2 \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 2e^{-i\pi/3}$

$z_2 = \frac{4i}{1-i} = \frac{(4i)^2}{1-i^2} = \frac{1+2i-1}{2} = i = e^{i\pi/2}$

$z_4 = \left(e^{i\pi/2}\right)^{16} = e^{i8\pi} = e^{i0} = 1$

$z_3 = -5 = 5e^{i\pi}$

④ $z_1 = 6\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = -3\sqrt{3} + 3i$

$z_2 = 18e^{i\pi/2} = 18e^{18 \cdot \frac{\pi}{2} + i\pi/2} = 18e^{i9\pi} e^{i\pi/2} = 18(-1)i = -18i$

⑤ $z^5 = -32 = 32e^{i\pi}$

les racines 5^{èmes} de $32e^{i\pi}$ sont les $z_k = 32^{1/5} e^{i\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right)}$ où $k=0,1,2,3,4$

$z_k = 2 e^{i\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right)}$

Réurrence

① $I_n = \int_0^1 1 \cdot (1-x^2)^n dx$

IdF: $u' = 1$ $u = x$
 $v = (1-x^2)^n$ $v' = n(1-x^2)^{n-1} \cdot (-2x)$

$$I_n = [x(1-x^2)^n]_0^1 + \int_0^1 n(1-x^2)^{n-1} 2x^2 dx$$

$$= 0 + 2n \int_0^1 (1-x^2)^{n-1} x^2 dx$$

$\hookrightarrow x^2 = 1 - (1-x^2)$

$$= 2n \int_0^1 (1-x^2)^{n-1} [1 - (1-x^2)] dx = 2n \int_0^1 (1-x^2)^{n-1} dx - 2n \int_0^1 (1-x^2)^n dx$$

$$I_n = 2n I_{n-1} - 2n I_n$$

$$\Rightarrow (2n+1)I_n = 2n I_{n-1} \Rightarrow \boxed{I_n = \frac{2n}{2n+1} I_{n-1}}$$

② Initial: pour $n=1$: $I_1 = \int_0^1 (1-x^2) dx = [x - \frac{x^3}{3}]_0^1 = 1 - \frac{1}{3} = \frac{2}{3} = \frac{2 \cdot 1}{3}$ OK

Hérédité: Je prends un $n \geq 1$. Je suppose que $I_n = \frac{2 \cdot 4 \dots (2n)}{3 \cdot 5 \dots (2n+1)}$

$$I_{n+1} = \frac{2(n+1)}{2n+3} I_n = \frac{2 \cdot 4 \dots (2n) \cdot 2(n+1)}{3 \cdot 5 \dots (2n+1) (2n+3)}$$

Ocei!

Conclusion: d'après le principe de récurrence: $\boxed{I_n = \frac{2 \cdot 4 \dots (2n)}{3 \cdot 5 \dots (2n+1)}, \forall n \in \mathbb{N}^*$

Produit des nombres pairs: $2 \cdot 4 \dots (2n) = 2 \cdot 1 \times 2 \cdot 2 \times \dots \times n \times 2 = 2^n \cdot n!$

Produit des nombres impairs: $1 \cdot 3 \cdot 5 \dots (2n+1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots (2n+1) \cdot (2n+1)}{2 \cdot 4 \dots (2n)} = \frac{(2n+1)!}{2^n n!}$

$$I_n = \frac{2^n n!}{\frac{(2n+1)!}{2^n n!}} = \boxed{\frac{(2^n n!)^2}{(2n+1)!}}$$

DLs et calculs de limites.

(1) $f(x) = \frac{\ln(x)}{1-x}$ changement $X = x - 1 \Rightarrow x = X + 1$

$$f(x) = \frac{\ln(1+X)}{1-(1+X)} = \frac{\ln(1+X)}{-X} = \frac{X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + o(X^4)}{-X} = -1 + \frac{X}{2} - \frac{X^2}{3} + \frac{X^3}{4} + o(X^3)$$

$$= -1 + \frac{(x-1)}{2} - \frac{(x-1)^2}{3} + \frac{(x-1)^3}{4} + o((x-1)^3)$$

tangente : $|y = -1 + \frac{x-1}{2}|$

$\Leftrightarrow 2y = -2 + x - 1 = x - 3$ $(D) : -x + 2y + 3 = 0$

La courbe est en dessous de la tangente car $-\frac{(x-1)^2}{3} \leq 0$

(2) $\tan(x) - \sin(x) = x + \frac{x^3}{3} + o(x^3) - \left(x - \frac{x^3}{6} + o(x^3)\right)$

$$= x^3 \left(\frac{1}{3} + \frac{1}{6}\right) + o(x^3) = x^3 \frac{2}{6} + o(x^3) = \frac{x^3}{2} + o(x^3)$$

$$x(2x^2 - \ln(\cos(3x))) = x(2x^2 - \ln(1 - \frac{9x^2}{2} + o(x^2)))$$

$$= x(2x^2 + \frac{9x^2}{2} + o(x^2)) = x(\frac{13x^2}{2} + o(x^2)) = \frac{13}{2}x^3 + o(x^3)$$

$$\frac{x(2x^2 - \ln(\cos(3x)))}{\tan x - \sin x} = \frac{\frac{13}{2}x^3 + o(x^3)}{\frac{x^3}{2} + o(x^3)} = \frac{13(\frac{1}{2} + o(1))}{\frac{1}{2} + o(1)} \xrightarrow{0} \frac{13/2}{1/2} = 13$$

(3) $\sqrt{x+x^2} \underset{x \rightarrow +\infty}{\sim} \sqrt{x^2} = x$ car $\frac{\sqrt{x+x^2}}{x} = \sqrt{\frac{x+x^2}{x^2}} = \sqrt{1+\frac{1}{x}} \xrightarrow{x \rightarrow +\infty} \sqrt{1} = 1$

$\sqrt{x+x^2} \underset{x \rightarrow 0}{\sim} \sqrt{x}$ car $\frac{\sqrt{x+x^2}}{\sqrt{x}} = \sqrt{1+x} \xrightarrow{x \rightarrow 0} \sqrt{1} = 1$

$f(x) = \sqrt{-x^2+3x-2}$ chgt $X = x - 1 \quad x = X + 1$

$$= \sqrt{-(1+2X+X^2)+3(1+X)-2} = \sqrt{-X-X^2} \underset{x \rightarrow 0}{\sim} \sqrt{-X}$$

donc $\sqrt{-x^2+3x-2} \underset{x \rightarrow 1}{\sim} \sqrt{x-1}$

continue. On appelle f cette fonction.

$\lim_{x \rightarrow 0} f(x) = 13$. Cette limite est finie.

On peut donc prolonger f par continuité en 0. On pose $f(0) = 13$