A Godunov-type method for the seven-equation model of compressible two-phase flow

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ARTICLE INFO

Article history:
Received 19 October 2010
Received in revised form 20 April 2011
Accepted 6 October 2011
Available online 18 October 2011

Keywords:
Godunov-type scheme
Approximate Riemann Solver
Seven equation two-phase flow model
Asymptotic-preserving scheme
Drift-flux model

ABSTRACT

We are interested in the numerical approximation of the solutions of the compressible seven-equation two-phase flow model. We propose a numerical strategy based on the derivation of a simple, accurate and explicit approximate Riemann solver. The source terms associated with the external forces and the drag force are included in the definition of the Riemann problem, and thus receive an upwind treatment. The objective is to try to preserve, at the numerical level, the asymptotic property of the solutions of the model to behave like the solutions of a drift-flux model with an algebraic closure law when the source terms are stiff. Numerical simulations and comparisons with other strategies are proposed.

One of the main features of this model is to involve two distinct velocities \( u_1 \) and \( u_2 \) and two pressures \( p_1 \) and \( p_2 \) associated with the two phases. It is actually more common, at least in the meantime and in the nuclear industry, to use two fluid-one pressure models where both phases share the same pressure \( p = p_1 = p_2 \). This approach is justified by the very short time-scale linked with the phenomenon of relaxation of the two pressures towards an equilibrium. However, the partial differential equations system corresponding to this assumption lacks some good mathematical properties: eigenvalues of the Jacobian matrix are not always real but may take complex values. The Cauchy problem for this system is then generally ill-posed. On the contrary, the two fluid-two pressure seven-equation model admits systematically seven real eigenvalues and is seen to have a basis of right eigenvectors, at least when resonance does not occur that will be considered throughout the present paper. The hyperbolicity property thus makes the two-fluid two-pressure approach very attractive in comparison to models using a pressure equilibrium assumption.

From a numerical point of view, the seven-equation model is not easy to deal with for several reasons. The first difficulty is related to the large size of the model: it is made of seven equations in one space dimension. As an immediate consequence, the Riemann problem associated with the convective part involves a large number of intermediate states (six generally speaking) and is difficult to determine or even approximate, and so is the derivation of Godunov-type method. The second difficulty comes from the presence of nonconservative products in the governing equations.

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doi:10.1016/j.compfluid.2011.10.004
and more precisely the fact that generally speaking, the model does not admit any equivalent conservation form (actually, the nonconservative terms vanish in the very particular situation where the void fractions $\alpha_k$ are locally constant in space and the structure of the model becomes the one of two (decoupled) classical gas dynamics systems). Note however that in this paper the nonconservative products will be associated with a linearly degenerate characteristic field so that by contrast with nonconservative products appearing in shocks (see for instance LeFloch [32,33]), discrete solutions are not expected to strongly depend on the underlying numerical viscosity (see Guillemaud [24]). Difficulties may arise but they are linked with the appearance of resonance phenomena (see Andrianov and Warnecke [5], Andrianov [3], etc.). At last but not least, the closure relations associated with the pressure laws $p_k$, $k = 1, 2$ may be strongly nonlinear in practice which renders even more difficult the derivation of exact or approximate Riemann solvers. A lot of papers are devoted to the numerical resolution of two-fluid two-pressure models and the question of how to discretize the nonconservative terms. In particular, the following is certainly not exhaustive. Let us first mention that Saurel and Abgrall [37] and Andrianov et al. [4] for instance (see also Saurel and Lemetayer [39] for a multidimensional framework) take into account the nonconservative terms by means of a free streaming physical condition associated with uniform velocity and pressure profiles. The discretization technique of [37] is improved by the same authors in [38]. Then, in Andrianov and Warnecke [5] and Schwendeman et al. [40], the common objective is to get exact solutions for the Riemann problem of the model. The approach is inverse in [5] in the sense that the initial left and right states are obtained as function of the intermediate states of the solution. On the contrary, a direct iterative approach is used in [40] leading to exact solutions of the Riemann problem for any initial left and right states. See also the recent work of Deledicque and Papalexandris [15]. Another direct approach is proposed in Castro and Toro [9]. In this work the authors propose to solve the Riemann problem approximately assuming that all the nonlinear characteristic fields are associated with rarefaction waves. More recently, Tokareva and Toro propose in [42] a HLLC-type approximate Riemann solver which takes into account all the seven waves that are naturally present in the model, and that can be seen as a similar but faster approach in comparison to the exact solver proposed in [40]. Finally, all these (approximate or exact) solutions are used to develop a Godunov-type method. At last, other finite volume methods have been used. For instance in Gallouët et al. [20] (see also Guillemaud [24]), the approximation of the convective terms of the system is based on the Rusanov scheme [36] and the so-called VF Roe-ncv scheme [8], these strategies being adapted to the nonconservative framework. In Munkejord [34] and Karni et al. [30], the authors use Roe-type schemes.

Many of the above mentioned methods are only devoted to the convective part of the seven-equation model, focusing in addition on a specific choice of the interface velocity $u_I$ naturally present in the governing equations. They are often based on an exact or approximate resolution of the complete Riemann problem, leading to fully nonlinear algorithms. In this context, our objective here is to derive the simplest possible approximate Riemann solver for the seven-equation model, which is accurate, explicit and well-adapted for the nuclear industry. By construction, it will be able to deal with any admissible interface velocity $u_I$ and any (possibly strongly nonlinear) closure relations for the pressure laws $p_k$, $k = 1, 2$. It will also enjoy several pleasant stability properties. Importantly, the definition of our approximate Riemann solver will also include the usual source terms associated with the gravity field and the drag force. The objective is to propose an upward treatment of the sources, so as to preserve the asymptotic behavior of the solutions in the asymptotic regime of relaxation time-scales tending to zero (see Section 2.3 for details). Proving the so-called asymptotic-preserving property of the proposed method unfortunately still remains an open problem. In that sense, this study can be understood as a first attempt to design asymptotic-preserving schemes for the seven-equation model.

The outline of the paper is as follows. Section 2 introduces some notations, gives the governing equations of the two-fluid two-pressure model and states its basic properties. Section 3 proposes a rough description of the whole numerical scheme. Next, Section 4 describes the approximate Riemann solver and Section 5 details the numerical scheme. At last, Section 6 gives the stability properties satisfied by the scheme and states some open problems. Section 7 is devoted to numerical experiments.

### 2. The governing equations and some mathematical properties

#### 2.1. Model

In one space dimension, the model under consideration reads as follows:

$$\begin{align*}
\frac{\partial}{\partial t} \alpha_k + \frac{\partial}{\partial x} \alpha_k u_k &= \Theta(p_k - p), \\
\frac{\partial}{\partial t} (\alpha_k Q_k) + \frac{\partial}{\partial x} (\alpha_k Q_k u_k) &= 0, \\
\frac{\partial}{\partial t} (\alpha_k Q_k u_k) + \frac{\partial}{\partial x} \left( \alpha_k (Q_k u_k^2 + p_k) \right) - p_k \frac{\partial}{\partial x} = \alpha_k Q_k g - K(u_k - u), \\
\frac{\partial}{\partial t} (\alpha_k Q_k \epsilon_k) + \frac{\partial}{\partial x} \left( \alpha_k (Q_k \epsilon_k + p_k) u_k \right) - p_k u_k \frac{\partial}{\partial x} &= \alpha_k Q_k g - p_k \Theta(p_k - p) - u_k K(u_k - u) \\
&= \alpha_k Q_k g - p_k \Theta(p_k - p) - u_k K(u_k - u)
\end{align*}$$

(1)

where $k, l = 1, 2$, $l \neq k$ with the constraint

$$\alpha_1 + \alpha_2 = 1,$$

(2)

meaning that the fluids are immisible. In the above equations, $\alpha_k$, $Q_k$, $u_k$, $\epsilon_k$ and $p_k$ denote the volume fraction, density, velocity, specific total energy and pressure of the phase $k$, $k = 1, 2$, while $g$ stands for the gravity acceleration. The structure of the convective part of system (1) is the one of two gas dynamics systems for each phase, coupled with a transport equation with velocity $u_k$ for the void fraction $\alpha_k$. We note however that non-conservative products involving the interfacial pressure $p_k$ and velocity $u_k$ (to be made precise later on) and the space derivative of the void fractions $\alpha_k$ are naturally present in the equations. These terms act as coupling terms in the evolution of the two phases in the sense that if they were not present (in case $\frac{\partial}{\partial x} = 0$ for instance), the phase evolutions would decouple and the usual conservative form of the gas dynamics equations would be recovered. In the general setting, we thus notice that system (1) is of the general nonconservative form

$$\frac{\partial}{\partial t} F(U) + \frac{\partial}{\partial x} B(U) \cdot \frac{\partial}{\partial x} U = S(U),$$

(3)

where

$$U = \begin{pmatrix} \alpha_1 \\ U_1 \\ U_2 \\ \alpha_2 U_2 \end{pmatrix}, \quad U_k = \begin{pmatrix} \alpha_k Q_k \\ \alpha_k Q_k u_k \\ \alpha_k Q_k \epsilon_k \end{pmatrix},$$

$$F(U) = \begin{pmatrix} 0 \\ F_1(U_1) \\ F_2(U_2) \end{pmatrix}, \quad F_k(U_k) = \begin{pmatrix} \alpha_k Q_k u_k \\ \alpha_k (Q_k u_k^2 + p_k) \\ \alpha_k (Q_k \epsilon_k + p_k) u_k \end{pmatrix},$$

(5)

$$S(U) = \begin{pmatrix} \Theta(p_1 - p_2) \\ S_1(U) \\ S_2(U) \end{pmatrix},$$

(6)
\[ S_k(U) = \begin{pmatrix} 0 \\ x_k Q_k g - K(u_k - u_i) \\ x_k Q_k \nu k - p_i \theta(p_k - p_i) - u_k K(u_k - u_i) \end{pmatrix}, \]

\[ B(U) = \begin{pmatrix} u_i \\ 0 \\ -p_i \\ -p_i u_i \\ p_i \\ 0 \end{pmatrix} : 0. \]

2.2. Closure relations

In order to close the system, we supplement (1) and (2) with closure relations that we now describe in details. It amounts to define the pressure laws \( p_k \), the pressure relaxation coefficient \( \theta \), the drag force coefficient \( K \), and the interfacial velocity \( u_i \) and pressure \( p_i \). We note in particular that we have decided to neglect the dissipation terms due to laminar or turbulent viscosity and heat conduction. Moreover, for the sake of simplicity, phase changes and convection terms due to laminar or turbulent viscosity and heat conduction between the two phases are not taken into account.

On the one hand, each phase is provided with an equation of state

\[ p_k = p_k(q_k, \epsilon_k), \quad k = 1, 2, \]

where \( \epsilon_k = \epsilon_i - \frac{u_i^2}{2} \) is the specific internal energy. On the other hand \( \theta \) will be typically chosen of the form

\[ \theta = \frac{1}{\tau_p} \frac{x_k x_2}{p_k + p_i}, \]

where \( \tau_p \) is the pressure relaxation time-scale, while \( K \) will be typically chosen of the form

\[ K = \frac{1}{8} C_{\text{dTm}} \rho_i |u_i - u_2|. \]

in the numerical simulations (see Section 7 for the definition of the related coefficients). Other choices are of course possible, we refer for instance the reader to Ishii and Zuber [27].

It remains to specify \( u_i \) and \( p_i \). With this in mind, we follow [20] and first observe that the eigenvalues of the Jacobian matrix \( F(U) + B(U) \) of (1) are always real and given by \( u_i, u_2, u_i \pm c_k, \) \( k = 1, 2, \) where \( c_k \) denotes the sound speed of the phase \( k \). The system (1) turns out to be typically weakly hyperbolic since the hyperbolicity property itself fails when \( u_i = u_2 \pm c_k \) for some index \( k \), i.e., resonance occurs. When the system (1) is hyperbolic, one can easily check that for \( k = 1, 2 \), and similarly to the classical gas dynamics equations, the characteristic fields associated with the eigenvalues \( u_i \pm c_k \) are genuinely nonlinear while the one associated with \( u_i \) is linearly degenerate. As far as the characteristic field associated with \( u_i \) is concerned, it is usually required to be linearly degenerate in practice. This property holds as soon as (see again [20])

\[ u_i = \beta u_1 + (1 - \beta) u_2, \quad \beta = \frac{x_2 x_1}{x_2 x_1 + (1 - \chi x_2)}, \]

where \( \chi \in [0, 1] \) is a constant, which allows to define the interfacial velocity \( u_i \). The usual choices for \( \chi \) are 0, \( \frac{1}{2} \), 1. Regarding the interfacial pressure \( p_i \), we set

\[ p_i = \mu \rho_1 + (1 - \mu) p_2, \quad \mu = \mu(U) \in [0, 1]. \]

The choice of the coefficient \( \mu \) is based on entropy considerations. If we denote by \( s_k = s_k(q_k, \epsilon_k) \) the specific entropy of the phase \( k \), it is an easy matter to check that any sufficiently smooth solution of (1) satisfies

\[ \left\{ \begin{array}{l}
\frac{\partial}{\partial t} \left( \sum_{k=1}^{7} x_k Q_k s_k \right) + \frac{\partial}{\partial x} \left( \sum_{k=1}^{7} x_k Q_k u_k s_k \right) + \sum_{k=1}^{7} \frac{1}{T_k} (p_k - p_i) (u_k - u_i) \frac{\partial x_k}{\partial x} \\
= \sum_{k=1}^{7} \frac{1}{T_k} \left[ K(u_k - u_i)(u_k - u_i) + \theta(p_k - p_i) (p_k - p_i) \right],
\end{array} \right. \]

where \( T_k \) is the temperature of the phase \( k \). Since by (12) and (13)

\[ \left\{ \begin{array}{l}
\left( 1 - \frac{\beta}{T_1} + \frac{\beta}{T_2} \right) K(u_2 - u_i)^2 + \left( \frac{1 - \mu}{T_1} + \frac{\mu}{T_2} \right) \theta(p_2 - p_i)^2 \geq 0,
\end{array} \right. \]

provided that we have

\[ \frac{1}{T_1} (p_i - p_1)(u_1 - u_i) - \frac{1}{T_2} (p_i - p_2)(u_2 - u_i) = 0, \]

which amounts to choose

\[ \mu = \mu(\beta) = \frac{1 - \beta}{\beta T_1 + (1 - \beta) T_2}. \]

Then, given a monotonically decreasing \( C^1 \) function \( \phi = \phi(s) \), we set

\[ \eta = \eta(U) = \sum_{k=1}^{7} x_k Q_k \phi(s_k) u_k. \]

Under the condition (12), we have for any sufficiently smooth solution of (1)

\[ \left\{ \begin{array}{l}
\frac{\partial \eta}{\partial t} + \frac{\partial q}{\partial x} = \left( \frac{1 - \mu}{T_1} + \frac{\mu}{T_2} \right) K(u_2 - u_i)^2 \\
+ \left( \frac{1 - \beta}{T_1} + \frac{\beta}{T_2} \right) \theta(p_2 - p_i)^2 \leq 0,
\end{array} \right. \]

so that \( (\eta, q) \) is a mathematical entropy pair if in addition the function \( \phi \) is convex. A shock solution of (1) associated with a nonlinear characteristic field should then satisfy the entropy inequality

\[ \left\{ \begin{array}{l}
\frac{\partial \eta}{\partial t} + \frac{\partial q}{\partial x} \leq \left( \frac{1 - \mu}{T_1} + \frac{\mu}{T_2} \right) K(u_2 - u_i)^2 \\
+ \left( \frac{1 - \beta}{T_1} + \frac{\beta}{T_2} \right) \theta(p_2 - p_i)^2 \leq 0,
\end{array} \right. \]

in the distributional sense. Note that the void fractions being transported with the velocity \( u_i \), they are naturally constant across a shock solution. Then, the non conservative products in the momentum and energy equations vanish and the system becomes conservative across these waves. Thus, entropic shock solutions can actually be defined in the usual distributional sense.

As far as discontinuities associated with linearly degenerate characteristic fields are concerned, they are naturally parametrized thanks to a set of Riemann invariants. In particular, entropy inequality (17) reads in this case
\[
\frac{\partial \eta}{\partial t} + \frac{\partial q}{\partial x} = 0.
\]

We refer to [20] for more details on all these considerations. The system is now closed thanks to the relations (9)-(13) and (15).

2.3. Asymptotic properties

The objective of this section is to mention some asymptotic properties satisfied by the solutions of (1). For that, we assume that the time-scales involved in the pressure relaxation and drag force coefficients are much smaller than the other characteristic time-scales of the system. More precisely, we assume that

\[
\Theta(U) = \frac{\theta(U)}{\varepsilon^2}, \quad K(U) = \Lambda(U)\|u_2 - u_1\|, \quad \Lambda(U) = \frac{j(U)}{\varepsilon^2},
\]

for a small parameter \(\varepsilon > 0\) which implies that

\[
p_2 - p_1 = O(\varepsilon^2), \quad u_2 - u_1 = O(\varepsilon).
\]

In other words, this means that the pressures relax to the equilibrium twice faster than the velocities, see [28]. Then, one can prove as \(\varepsilon\) tends to 0 that (1) admits an asymptotic limit, the so-called differential drift-flux limit which we now briefly recall for the sake of completeness. We refer for instance the reader to [2,14,23], and the references therein for more details. Let us define the following mixture quantities: density \(\varrho\), velocity \(u\), pressure \(p\) and total energy \(e\) by

\[
\varrho = \sum_{k=1}^{2} \varrho_k q_k, \quad \varrho u = \sum_{k=1}^{2} \varrho_k q_k u_k, \quad p = \sum_{k=1}^{2} \varrho_k p_k, \quad qe = \sum_{k=1}^{2} \varrho_k \varrho_k e_k.
\]

(20)

We introduce in addition the relative velocity \(u_r\), and the mass fraction \(Y\) of the phase 2:

\[
u_r = u_2 - u_1, \quad Y = \frac{\varrho_2 q_2}{\varrho}.
\]

Then, following the classical Chapman–Enskog method, it can be proved that a first-order approximation with respect to \(\varepsilon\) of (1) is given by the following differential drift-flux model:

\[
\begin{align*}
\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho u}{\partial x} &= 0, \\
\frac{\partial \varrho Y}{\partial t} + \frac{\partial \varrho u Y}{\partial x} + Y(1 - Y)u_2 &= 0, \\
\frac{\partial \varrho u}{\partial t} + \frac{\partial \varrho u^2}{\partial x} + p + Y(1 - Y)u_2^2 &= \varrho q, \\
\frac{\partialqe}{\partial t} + \frac{\partial (qe + p) u + Y(1 - Y)u_2^2 u}{\partial x} &= \varrho g u, 
\end{align*}
\]

where the relative velocity \(u_r\) is given by the differential drift law

\[
u_r | u_r = \frac{\varrho Y(1 - Y)}{A} \left(1 - \frac{1}{\varrho_1} \right) \frac{\partial \varrho}{\partial x}.
\]

(22)

Closure relations for \(\rho_1, \rho_2\) and \(p\) are not detailed here and we refer again to [14,23]. We note that (22) involves the drag-force coefficient but is independent of the pressure relaxation coefficient. It also depends on the unknown derivatives by the pressure gradient \(\frac{\partial \varrho}{\partial x}\). Of course, if we assume for instance that the pressure gradient is balanced by the gravity force, that is

\[
\frac{\partial \varrho}{\partial x} = \varrho g,
\]

as it is the case for permanent flows, (23) has to be replaced by the following algebraic (zeroth-order) drift law

\[
u_r|u_r = -\frac{\varrho Y(1 - Y)}{A} \left(1 - \frac{1}{\varrho_1} \right) \varrho g.
\]

(23)

We obtain in this case the classical drift-flux one-fluid model (see for instance [26]), which definitely proves a hierarchy between the drift-flux models and the seven-equation model.

This asymptotic property holds true at the continuum level. A challenging issue is to retain the validity of this asymptotic behaviour from a numerical point of view, by means of an asymptotic-preserving scheme for the seven-equation model. In other words, such a scheme is expected to provide a consistent approximation of the drift-flux model (22) and (23) in the asymptotic regime \(\varepsilon \to 0\).

The rest of the paper is precisely devoted to the derivation of a numerical scheme for the seven-equation model. It will be based on a splitting strategy between the pressure relaxation terms and the other ones, namely the convective terms and both the gravity and drag force terms. A particular attention will be paid to the latter terms for which we will propose an approximate Godunov method. In particular, we emphasize from now on that the sources will be included in the definition of the underlying approximate Riemann solver, and thus will receive an upwind treatment. The motivation to split the pressure relaxation terms comes from the very fast pressure relaxation time-scales to the equilibrium \(p_1 = p_2\) used in (19).

3. The principle of the numerical method

As already mentioned, we will use a splitting strategy to solve numerically the system (1). Let us first introduce a space step \(\Delta x\) and a time step \(\Delta t\), and set

\[x_i = i \Delta x, \quad i \in \mathbb{Z}, \quad t^n = n \Delta t, \quad n \in \mathbb{N}.\]

We denote by \(U^0\) an approximation of \(U(x_i, t^n)\). Then, starting from \(U^n = (U^n_i)_{i \in \mathbb{Z}}\), we propose to compute the approximate solution \(U^{n+1}\) at time \(t^{n+1}\) by means of the following two-step procedure.

(i) First step. We solve numerically the system

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial \varrho_k}{\partial t} + \frac{\partial \varrho_k u_k}{\partial x} = 0, \\
\frac{\partial \varrho_k Y}{\partial t} + \frac{\partial \varrho_k u Y}{\partial x} + Y(1 - Y)u_k &= 0, \\
\frac{\partial \varrho_k u}{\partial t} + \frac{\partial \varrho_k u^2}{\partial x} + p + Y(1 - Y)u_k^2 &= \varrho_k q, \\
\frac{\partial \nu_k e}{\partial t} + \frac{\partial (\nu_k e + p) u + Y(1 - Y)u_k^2 u}{\partial x} &= \varrho_k g u_k,
\end{array} \right.
\]

where \(\nu_k\) is the relative velocity of the phase \(k\), \(k = 1, 2\), \(A_k\) the Riemann solver, and \(\varrho_k\) the mass fraction of the phase \(k\).

(24)

\[
\begin{align*}
\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho u}{\partial x} &= 0, \\
\frac{\partial \varrho Y}{\partial t} + \frac{\partial \varrho u Y}{\partial x} + Y(1 - Y)u_2 &= 0, \\
\frac{\partial \varrho u}{\partial t} + \frac{\partial \varrho u^2}{\partial x} + p + Y(1 - Y)u_2^2 &= \varrho q, \\
\frac{\partialqe}{\partial t} + \frac{\partial (qe + p) u + Y(1 - Y)u_2^2 u}{\partial x} &= \varrho g u.
\end{align*}
\]

(25)

(obtained from (1) by setting \(\Theta = 0\) using \(U^0\) as initial condition. We denote by \(U^{n+1/2}\) the solution of (25) obtained at time \(t^{n+1/2}\). This will be achieved by means of a new Godunov-type method described in the next section, and which is able to handle the source terms (the drag force and gravity) in a consistent way in order to preserve the asymptotic properties of (1). Such an approach, leading to an upwind treatment of the sources, seems indeed natural if we refer for instance to [10]. In this paper, the authors consider the classical gas dynamics system with gravity and friction, and propose an asymptotic-preserving Godunov-type scheme (when the friction coefficient goes to infinity) including the sources in the derivation of the underlying approximate Riemann solver. Here, in some sense, the seven-equation can be understood as two gas dynamics system with friction and gravity coupled by an additional equation on the void fractions, which justifies our approach.

(ii) Second step. The pressure relaxation term are now taken into account: the numerical solution of (1) at time \(t^{n+1}\) is computed by solving the following ODE system

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial \varrho_k}{\partial t} + \frac{\partial \varrho_k u_k}{\partial x} = 0, \\
\frac{\partial \varrho_k Y}{\partial t} + \frac{\partial \varrho_k u Y}{\partial x} + Y(1 - Y)u_k &= 0, \\
\varrho_k \varrho_k u_k \frac{\partial \varrho_k}{\partial x} = p_i \Theta(p_k - p_i),
\end{array} \right.
\]

(26)
with the initial condition $U_0$. Actually, this system has already been analyzed and solved numerically in [20]. For reader’s convenience and for the sake of completeness, we briefly recall the main ideas of this strategy that we have used in practice, and refer the reader to [20] for the details. The main objective of the procedure is to preserve the positivity of the pressures $p_1$ and $p_2$ as well as the maximum principle $z_k \in [0, 1]$ for the void fractions.

Let us first focus on the void fractions. Using (10), the first equation of (26) equivalently writes

$$
\frac{1}{x_1 x_2} \frac{d}{dt} \left( \frac{x_1}{1-x_1} \right) = \frac{1}{t} \left( p_1 - p_2 \right)
$$

that is, since $x_1 + x_2 = 1$,

$$
\frac{d}{dt} \ln \left( \frac{x_1}{1-x_1} \right) = \frac{1}{t} \left( p_1 - p_2 \right).
$$

We thus get after integration

$$
\left( \frac{x_1}{1-x_1} \right)(t) = \left( \frac{x_1}{1-x_1} \right)^{n+1/2} \exp \left( \frac{1}{t} \int_0^t \left( p_1 - p_2 \right)(\tau)d\tau \right).
$$

This formula clearly provides us with a unique definition of $z_k(t) \in [0, 1]$ whatever be the proposed quadrature formula and provided that $(x_k)^{n+1/2} \in [0, 1]$, $k = 1, 2$.

Regarding the two pressures $p_1$ and $p_2$, we first write using (27)

$$
\frac{dp}{dt} = \frac{\partial p}{\partial p_k} \frac{dp_k}{dt} + \frac{\partial p}{\partial x_k} \frac{dx_k}{dt} = - \left( \frac{\partial p}{\partial p_k} \frac{\partial p_k}{\partial x_k} \frac{dx_k}{dt} \right) \frac{dp_k}{dt}, \quad k = 1, 2.
$$

To avoid cumbersome notations, we propose to set

$$
A_k = \left( \frac{\partial p}{\partial p_k} \right)_{x_k = x_k^0} \frac{p_1}{p_k} - \frac{\partial p}{\partial x_k} \frac{dx_k}{dt}, \quad k = 1, 2.
$$

Subtracting these two equations and using the first equation of (26) leads to

$$
\frac{d}{dt} (p_k - p_i) + \Theta(A_k - A_i)(p_k - p_i) = 0,
$$

which gives after integration

$$
(p_k - p_i)(t) = (p_k - p_i)^{n+1/2} \exp \left( - \int_0^t \Theta(A_k - A_i)(\tau)d\tau \right), \quad k = 1, 2.
$$

This first equation proves that $(p_k - p_i)(t)$ and $(p_k - p_i)^{n+1/2}$ keep the same sign for all $t$. Now dividing (29) by $p_k$ gives

$$
\frac{1}{p_k} \frac{d}{dt} \ln p_k + \frac{A_k}{p_k} \frac{d}{dt} \ln x_k = 0,
$$

and then

$$
\frac{1}{p_k} \frac{d}{dt} \ln p_k + \frac{A_k}{p_k} \frac{d}{dt} \ln x_k = 0.
$$

Integrating gives

$$
(p_1, p_2)(t) = (p_1, p_2)^{n+1/2} \exp \left( - \int_0^t \left( \sum_{k=1}^2 \frac{A_k}{p_k} \frac{d}{dt} \ln x_k \right)(\tau)d\tau \right).
$$

This second equation shows that $p_1$ and $p_2$ have necessarily the same sign provided that $(p_1, p_2)^{n+1/2} > 0$. Formulas (30) and (31) thus provide us with unique positive values of $p_1(t)$ and $p_2(t)$ for all time $t$, provided that $(p_1)^{n+1/2} > 0$ and $(p_2)^{n+1/2} > 0$.

On the ground of the above developments, the definition of the numerical solution to system (26) given in [20] is based on a natural discretization of

$$
\left( \begin{array}{c}
\frac{d}{dt} \left( \frac{x_1}{1-x_1} \right) \\
\frac{d}{dt} p_1 \\
\frac{d}{dt} p_2
\end{array} \right) = \left( \begin{array}{c}
\left( \frac{x_1}{1-x_1} \right)^{n+1/2} \exp \left( \frac{1}{t} \int_0^t \left( \frac{p_1 - p_2}{p_1 + p_2} \right)(\tau)d\tau \right) \\
\frac{1}{t} \left( p_1 - p_2 \right) \\
\frac{1}{t} \left( p_1 - p_2 \right)
\end{array} \right).
$$

This strategy clearly allows to define positive pressures $p_k$ and void fractions $x_k$ lying in $[0, 1]$. The partial densities also remain positive provided that they are at the end of the first step.

We now focus in the next section on the numerical approximation of the first step and more precisely on the derivation of a consistent approximate Riemann solver for (25).

4. The approximate Riemann solver

The system (25) is again of the general form (3) where $U, F(U)$ and $B(U)$ are still given by (4), (5) and (8) respectively while $S(U)$ is now defined by

$$
S(U) = \begin{pmatrix}
0 \\
S_1(U) \\
S_2(U)
\end{pmatrix}, \quad S_1(U) = \begin{pmatrix}
x_k g - K(u_k - u_i)
\end{pmatrix}, \quad S_2(U) = \begin{pmatrix}
0 \\
x_k g - K(u_k - u_i)
\end{pmatrix}.
$$

Our purpose is now to construct an approximate Riemann solver for the system (25).

4.1. Generalities

We first recall in this section a fairly general method of construction of Riemann solvers for a nonconservative systems with sources of $p$ equations of the form (3). More precisely, our main objective is to recall, following Gallice [17–19], the notion of consistency with the integral form of (25) of such an approximate Riemann solver.

We first introduce a simple Riemann solver of the form

$$
W_\Lambda \left( \begin{array}{c}
x_{l-1} \\
U_{l-1} \\
U_{l}
\end{array} \right) = \left( \begin{array}{c}
U_{l-1} \\
U_k \\
U_k
\end{array} \right), \quad \sigma_{k-1} < \frac{\theta}{\sigma_k}, \quad k = 2, \ldots, m,
$$

where $m$ is the number of waves in the system.

This consists of $m$ waves with speeds $\sigma_k = \sigma_k(U_k, U_{k-1})$, $1 \leq k \leq m$, and $m - 1$ intermediate states $U_{i,l} \leq k \leq m$, which may depend on $A = (\Delta x, \Delta t)$.

As it is natural, we will first require

$$
\lim_{A \to 0} W_\Lambda \left( \begin{array}{c}
x_{l-1} \\
U_{l-1} \\
U_{l}
\end{array} \right) = U.
$$

Let us now define the consistency in the integral sense.

4.1.1. Consistency with the integral form of (25)

We first observe that, under the CFL condition

$$
\max_{1 \leq i \leq m} \left| \sigma_i \frac{\Delta x}{\Delta t} \right| < \frac{1}{2},
$$

In the sequel, the subscript $\Lambda = (\Delta x, \Delta t)$ will indicate a dependence on $\Delta x$ and $\Delta t$. We now define the consistency in the integral sense.

$$
\lim_{A \to 0} W_\Lambda \left( \begin{array}{c}
x_{l-1} \\
U_{l-1} \\
U_{l}
\end{array} \right) = U.
$$

Let us now define the consistency in the integral sense.
we have
\[ \frac{1}{\Delta x} \int_{-\Delta x}^{\Delta x} f(U(x), \nabla U(x)) \, dx = \frac{1}{2} (U_L + U_R) - \frac{\Delta t}{\Delta x} \sum_{i=1}^{m} \sigma_i(U_{i-1} - U_i), \]
so that
\[ \int_{-\Delta t}^{\Delta t} \int_{-\Delta x}^{\Delta x} \frac{\partial}{\partial t} \mathbf{W}(x, U(x), \nabla U(x)) \, dx \, dt = -\Delta t \sum_{i=1}^{m} \sigma_i(U_{i-1} - U_i). \]

Next, integrating (25) in space and time leads to
\[ \int_{-\Delta t}^{\Delta t} \int_{-\Delta x}^{\Delta x} \mathbf{W}(x, U(x), \nabla U(x)) \, dx \, dt \]
\[ = \int_{-\Delta t}^{\Delta t} \int_{-\Delta x}^{\Delta x} \mathbf{S}(U(x), \nabla U(x)) \, dx \, dt. \]

which gives, focusing ourselves on simple Riemann solutions of the form (34),
\[ \int_{-\Delta t}^{\Delta t} \mathbf{W}(x, U(x), \nabla U(x)) \, dx \]
\[ = \frac{\Delta t}{2} (U_L + U_R) - \int_{-\Delta t}^{\Delta t} \int_{-\Delta x}^{\Delta x} \left\{ \frac{\partial}{\partial x} \mathbf{F}(U(x)) + \mathbf{B}(U(x)) \frac{\partial}{\partial x} \mathbf{W}(x, U(x), \nabla U(x)) \right\} \, dx \, dt. \]

Consistency in the integral sense in imposing
\[ \int_{-\Delta t}^{\Delta t} \int_{-\Delta x}^{\Delta x} \left\{ \frac{\partial}{\partial x} \mathbf{F}(U(x)) + \mathbf{B}(U(x)) \frac{\partial}{\partial x} \mathbf{W}(x, U(x), \nabla U(x)) \right\} \, dx \, dt \approx \Delta t \sum_{i=1}^{m} \sigma_i(U_{i-1} - U_i). \]

(37)

More precisely, the Riemann solver (34) is said to be consistent with (3) if there exists a \( p \times p \) matrix \( \mathbf{B}(U(x), \nabla U(x)) \) and a vector \( \mathbf{S}(U(x), \nabla U(x)) \) in \( \mathbb{R}^p \) with
\[ \lim_{U_{i-1} \to U \atop U_i \to U} \mathbf{B}(U, \nabla U) = \mathbf{B}(U), \]
\[ \lim_{U_{i-1} \to U \atop U_i \to U} \mathbf{S}(U, \nabla U) = \mathbf{S}(U), \]

such that under the CFL condition (36),
\[ \Delta t \mathbf{F} + \mathbf{B}_i(U(x), \nabla U(x)) \Delta U - \Delta x \mathbf{S}_i(U(x), \nabla U(x)) \]
\[ = \sum_{i=1}^{m} \sigma_i(U_{i-1} - U_i), \]

(40)

where as usual \( \Delta U = U_L - U_R \) and \( \Delta t \mathbf{F} = \mathbf{F}(U_L) - \mathbf{F}(U_R) \). Note that the left-hand side of (36) is obtained by an exact calculation of the left-hand side of (37) where the exact matrix \( \mathbf{B} \) and the exact vector \( \mathbf{S} \) have been replaced with \( \mathbf{B}_i \) and \( \mathbf{S}_i \). Observe also that we recover the usual definition of consistency associated with systems of conservation laws when the non-conservative products and the sources are not present in the model (see for instance [25]).

4.1.2. The Godunov-type method

The Godunov-type method associated with the simple Riemann solver (34) is then defined as usual by
\[ \left\{ \mathbf{U}_{i+1}^{n+1} \right\} = \left\{ \int_{-\Delta t}^{\Delta t} \mathbf{W}_i \left( \mathbf{x}, \mathbf{U}_i^{n+1} \right) \, dx \right\}. \]

(41)

Assuming that the Riemann solver (34) is consistent with (3) and setting
\[ \mathbf{G}_i(U_L, U_R) = \frac{1}{2} \left\{ \mathbf{F}(U_L) + \mathbf{F}(U_R) - \sum_{i=1}^{m} \sigma_i(U_{i-1} - U_i) \right\}, \]

(42)

one can check that, under the CFL condition (36), the associated Godunov-type method reads
\[ \left\{ \mathbf{U}_{i+1}^{n+1} \right\} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x} \left( \mathbf{G}_i^n - \mathbf{G}_{i+1}^n \right) \]
\[ - \frac{\Delta t}{2\Delta x} \left( \mathbf{B}_i^{n+1} (\mathbf{U}_i^n - \mathbf{U}_{i+1}^n) + \mathbf{B}_{i+1}^{n+1} (\mathbf{U}_{i+1}^n - \mathbf{U}_i^n) \right) + \frac{\Delta t}{2} \left( \mathbf{S}_i^{n+1} + \mathbf{S}_{i+1}^{n+1} \right). \]

(43)

where for all \( i \)
\[ \mathbf{G}_i^n = \mathbf{G}_i(U_i^n, \mathbf{U}_{i+1}^n), \quad \mathbf{B}_i^{n+1} = \mathbf{B}_i(U_i^n, \mathbf{U}_{i+1}^n), \]
\[ \mathbf{S}_i^{n+1} = \mathbf{S}_i(U_i^n). \]

Once the approximate Riemann solver is defined according to (40), the proposed Godunov-type method is then very classical. We note however that the non-conservative products and the sources are taken into account in the definition of consistency, and then clearly receive an upwind treatment in formula (43).

4.2. Construction of the Riemann solver

In order to construct an approximate Riemann solver for the system (25), it is first convenient to associate to (25) the linearized system
\[ \left\{ \frac{\partial}{\partial x} \mathbf{U}_i \right\} + \frac{\partial}{\partial t} \mathbf{U}_i = 0, \]
\[ \frac{\partial}{\partial x} \mathbf{U}_i + \frac{\partial}{\partial t} \mathbf{U}_i = 0, \]
\[ \frac{\partial}{\partial x} \mathbf{U}_i + \frac{\partial}{\partial t} \mathbf{U}_i = 0, \]

(45)

\( k, l = 1, 2, l \neq k \), which is obtained by replacing \( \mathbf{U}_i \) by \( \Delta \mathbf{U}_i \) and introducing a new variable \( \mathbf{U}_i \) which plays the role of a Lagrangian sound speed. This approach allows to relax the possibly strong non-linearities involved in the pressure laws and is similar to the one used for instance in the well-known HLLC approach, or in the relaxation schemes (see for instance [71, 11], and the references therein). Setting
\[ \mathbf{V}_i = (\mathbf{U}_i, \mathbf{U}_i^{(T)})^T, \quad k = 1, 2, \]

(46)

we assume as given the pair \( \mathbf{V}_{k+1}, \mathbf{V}_{k+1} \) with
\[ \mathbf{P}_{k+1} = \mathbf{P}_{k+1}(\mathbf{V}_{k+1}, \mathbf{V}_{k+1}), \quad \lambda = L, R. \]

Defining \( \mathbf{P}_{k+1} \) at its equilibrium value \( \mathbf{P}_{k+1} \) for the left and right states of the Riemann initial condition implies that the consistency relations (40) are identical, the system under consideration being linearized (i.e. (45)) or not (i.e. (25)).

We construct the Riemann solver as follows.

4.2.1. First notations and form of the approximate Riemann solver

We first introduce an approximation
\[ \mathbf{U}_i = \mathbf{U}_i / |\mathbf{V}_i| \cdot \mathbf{v}_i, \quad \mathbf{v}_i = \frac{\mathbf{V}_{i+1} - \mathbf{V}_{i-1}}{2\lambda}, \quad \lambda = L, R. \]

of the interfacial velocity \( \mathbf{v}_i \) which will be determined in the sequel. Then the first and last Eq. (45) lead us to take
\[ \mathbf{v}_i = \frac{\mathbf{v}_i}{\lambda}, \quad \lambda = L, R, \]

(48)

and
\[ \mathbf{C}_L = \frac{\mathbf{V}_L}{\lambda}, \quad \mathbf{C}_R = \frac{\mathbf{V}_R}{\lambda}. \]

(49)
For $k = 1, 2$, we next introduce a function $V_{k,i}$ of the form
\[
V_{k,i} = \begin{cases}
\frac{1}{\sigma_k} < \sigma_{k,1}, & V_{k,L}, \\
\sigma_{k,1} < \frac{1}{\sigma_k} < \sigma_{k,2}, & V_{k,L}; \\
\sigma_{k,2} < \frac{1}{\sigma_k} < \sigma_{k,3}, & V_{k,R}, \\
\frac{1}{\sigma_k} > \sigma_{k,3}, & V_{k,R},
\end{cases}
\]
where the wave speed estimates $\sigma_{k,1} < \sigma_{k,2} < \sigma_{k,3}$ are given by
\[
\begin{aligned}
\sigma_{k,1} &= \frac{c_k}{\tilde{c}_k}, \\
\sigma_{k,2} &= \frac{c_k}{\tilde{c}_k}, \\
\sigma_{k,3} &= \frac{c_k}{\tilde{c}_k}.
\end{aligned}
\]
Note that by (48) and (49) we have $\sigma_{k,1} = \frac{c_k}{\tilde{c}_k}$, $C_{k,1} = C_{k,2}$, $\lambda = L, R,$ (52)

Now, if we set using obvious notations
\[
U_{k,i} = U_k(V_{k,i}), \quad U'_{k,i} = U_k(V_{k,i}), \quad U_j = \begin{pmatrix} \tilde{c}_k \\ U_{1,j} \\ U_{2,j} \end{pmatrix}, \quad \lambda = L, R,
\]
the Riemann solver is taken in the form
\[
W_{k,i} = \begin{pmatrix} U_{k,1} \\ U_{k,1} \\ U_{k,1} \\ U_{k,R} \\ U_{k,R} \\ U_{k,R} \\ U_{k,R} \\
\end{pmatrix},
\]
with
\[
W_{k,i} = \begin{pmatrix} U_{k,1} \\ U_{k,1} \\ U_{k,1} \\ U_{k,R} \\ U_{k,R} \\ U_{k,R} \\ U_{k,R} \\
\end{pmatrix},
\]
Indeed, in (50) (resp. in (55)), $V_{k,i}$ (resp. $U'_{k,i}$) and $\sigma_2 = \frac{c_k}{\tilde{c}_k}$ will depend on $\lambda = (\Delta x, \Delta t)$.

We next consider the momentum jump relations for the mass conservation equation at each wave with speed $\sigma_k$, $1 \leq k \leq 3$:
\[
\begin{aligned}
\sigma_{k,1} (\tilde{c}_k u - \tilde{c}_k u) &= \tilde{c}_k u - \tilde{c}_k u, \\
\sigma_{k,2} (\tilde{c}_k u - \tilde{c}_k u) &= \tilde{c}_k u - \tilde{c}_k u, \\
\sigma_{k,3} (\tilde{c}_k u - \tilde{c}_k u) &= \tilde{c}_k u - \tilde{c}_k u.
\end{aligned}
\]
By summing the above Eq. (59), we obtain that the first component of the consistency condition (58) is trivially satisfied. Together with the expressions (51) of $\sigma_k$, $1 \leq k \leq 3$ the first and third relations (59) yield
\[
\begin{aligned}
\sigma_{k,1} = \frac{c_k}{\tilde{c}_k} u - \frac{c_k}{\tilde{c}_k} u + \frac{c_k}{\tilde{c}_k} u, \\
\sigma_{k,3} = \frac{c_k}{\tilde{c}_k} u + \frac{c_k}{\tilde{c}_k} u + \frac{c_k}{\tilde{c}_k} u.
\end{aligned}
\]
While the second relation (59) reads
\[
\sigma_{k,2} = \frac{c_k}{\tilde{c}_k} u - \frac{c_k}{\tilde{c}_k} u = \frac{c_k}{\tilde{c}_k} u - \frac{c_k}{\tilde{c}_k} u.
\]

4.2.2. The consistency relations

It thus remains to define the wave speed $\tilde{c}_k$ and the intermediate states $V_{k,i}$, $k = 1, 2$, $\lambda = L, R$, in such a way that the approximate Riemann solver is consistent with the integral form of system (25), or equivalently that the consistency relations (40) are valid. This first necessities to define the matrix $B_k$ and the vector $S_k$. We propose to simply set
\[
S_k = \begin{pmatrix} 0 \\ S_{k,1} \\ S_{k,1} \end{pmatrix}, \quad S_{k,1} = \begin{pmatrix} \tilde{c}_k - K(u_k - u_l) \\ \tilde{c}_k - K(u_k - u_l) \\ \tilde{c}_k - K(u_k - u_l) \end{pmatrix}
\]
and
\[
B_k = \begin{pmatrix} \tilde{c}_k - K(u_k - u_l) \\ \tilde{c}_k - K(u_k - u_l) \\ \tilde{c}_k - K(u_k - u_l) \end{pmatrix}
\]
\[
\left\{ \begin{array}{l}
\frac{d}{dt}(z_k u_k^2) + \Pi_k - \bar{p}_k \Delta z_k - \Delta \left( z_k \bar{q}_k g - K(u_k - u) \right)
= \sigma_{k,1} z_k \left( \frac{z_k}{\bar{q}_k} u_k^2 \right) + \bar{u}_k (z_k \frac{z_k}{\bar{q}_k} u_k^2 - z_k \bar{q}_k u_k^2) \\
+ \sigma_{k,2} z_k (\bar{q}_k u_k - z_k \bar{q}_k u_k^2)
\end{array} \right.
\]

Using (62), we obtain
\[
\left\{ \begin{array}{l}
\bar{u}_k (z_k \bar{q}_k u_k - \bar{q}_k z_k u_k^2) \\
= \sigma_{k,1} z_k \left( \frac{z_k}{\bar{q}_k} u_k^2 \right) + \bar{u}_k (z_k \frac{z_k}{\bar{q}_k} u_k^2 - z_k \bar{q}_k u_k^2) \\
+ \sigma_{k,2} z_k (\bar{q}_k u_k - z_k \bar{q}_k u_k^2) \\
- \Delta \left( z_k \bar{q}_k g - K(u_k - u) \right).
\end{array} \right.
\]

In fact, this equation may be viewed as a Rankine–Hugoniot jump relation when crossing the coupling wave whose speed is \( u^i \) provided that the source term acts as a Dirac mass located along that wave.

Let us now arrange these relations in a more convenient way. By using the expression (60) of \( \sigma_{k,1} \) the first Eq. (62) becomes
\[
\left\{ \begin{array}{l}
\bar{u}_k \left( \frac{z_k}{\bar{q}_k} u_k^2 \right) + \bar{u}_k (z_k \frac{z_k}{\bar{q}_k} u_k^2 - z_k \bar{q}_k u_k^2) \\
+ \sigma_{k,2} z_k (\bar{q}_k u_k - z_k \bar{q}_k u_k^2) \\
- \Delta \left( z_k \bar{q}_k g - K(u_k - u) \right).
\end{array} \right.
\]

or
\[
\Pi_k = \Pi_k + \alpha_k C_k \left( u_k^2 - u_k \right).
\]

Similarly, the second Eq. (62) reads
\[
\Pi_k = \Pi_k + \alpha_k C_k \left( u_k^2 - u_k \right).
\]

On the other hand, by using (61), Eq. (63) becomes
\[
\left\{ \begin{array}{l}
\Pi_k = \Pi_k + \alpha_k C_k \left( u_k^2 - u_k \right) \\
- \Delta \left( z_k \bar{q}_k g - K(u_k - u) \right).
\end{array} \right.
\]

Then, replacing in (66) \( \Pi_k \) and \( \Pi_k \) by their expressions (64) and (65) gives
\[
\left\{ \begin{array}{l}
\Delta \Pi_k - \bar{p}_k \Delta z_k = j_k \left( u_k^2 - u_k \right) + \bar{p}_k (\bar{z}_k - z_k) + \Delta \left( z_k \bar{q}_k g - K(u_k - u) \right),
\end{array} \right.
\]

where here and in all the sequel \( \varphi_k = \frac{1}{2} (\varphi_k + \varphi_k^i) \) will denote the arithmetic average of any pair of quantities \( (\varphi_k, \varphi_k^i) \). Notice that (60) and (61) yield
\[
\left\{ \begin{array}{l}
\bar{u}_k \left( \frac{z_k}{\bar{q}_k} u_k^2 \right) + \bar{u}_k (z_k \frac{z_k}{\bar{q}_k} u_k^2 - z_k \bar{q}_k u_k^2) \\
+ \sigma_{k,2} z_k (\bar{q}_k u_k - z_k \bar{q}_k u_k^2) \\
- \Delta \left( z_k \bar{q}_k g - K(u_k - u) \right).
\end{array} \right.
\]

and
\[
\left\{ \begin{array}{l}
\bar{u}_k \left( \frac{z_k}{\bar{q}_k} u_k^2 \right) + \bar{u}_k (z_k \frac{z_k}{\bar{q}_k} u_k^2 - z_k \bar{q}_k u_k^2) \\
+ \sigma_{k,2} z_k (\bar{q}_k u_k - z_k \bar{q}_k u_k^2) \\
- \Delta \left( z_k \bar{q}_k g - K(u_k - u) \right).
\end{array} \right.
\]

Then, taking (47) into account, we obtain
\[
\left\{ \begin{array}{l}
\Delta (\alpha_k u_k) - \bar{p}_k \Delta z_k = \bar{u}_k \left( \frac{z_k}{\bar{q}_k} u_k^2 \right) + \sigma_{k,2} z_k (\bar{q}_k u_k - z_k \bar{q}_k u_k^2) \\
+ \Delta \left( z_k \bar{q}_k g - K(u_k - u) \right).
\end{array} \right.
\]

The interfacial velocity \( u^i \). Let us now define the approximate interfacial velocity \( u^i \). The definition (12) of \( u^i \) leads us to set
\[
u^i = \bar{u}_i \left( \frac{z_k}{\bar{q}_k} u_k^2 \right) + \bar{u}_i (z_k \frac{z_k}{\bar{q}_k} u_k^2 - z_k \bar{q}_k u_k^2) \\
\text{or equivalently}
\]

\[
u^i = \bar{u}_i \left( \frac{z_k}{\bar{q}_k} u_k^2 \right) + \bar{u}_i (z_k \frac{z_k}{\bar{q}_k} u_k^2 - z_k \bar{q}_k u_k^2) \\
\text{This amounts to set}
\]

\[
\nu^1 + (1 - \nu^2)_j = 0.
\]

The energy equations and the calculation of \( e^i_k \). For \( k = 1, 2 \), it remains to determine the pair \( (e^i_k, e^i_k^i) \). First, the third component of the consistency relation (58) reads
\[
\Delta (\alpha_k u_k) - \bar{p}_k \Delta z_k = \bar{u}_k \left( \frac{z_k}{\bar{q}_k} u_k^2 \right) + \sigma_{k,2} z_k (\bar{q}_k u_k - z_k \bar{q}_k u_k^2) \\
+ \Delta \left( z_k \bar{q}_k g - K(u_k - u) \right).
\]

\[
\frac{1}{
\text{2. Associated with the phase } k = 1 \text{ for instance.}
\text{3. Provided that the waves remain well ordered as they should, i.e. } \sigma_{k,1} > \sigma_{k,1},
\text{for } k = 1, 2.
\]
Using again (60) and (61) together with (59), the consistency relation (74) becomes
\[
\Delta t \left( H_t(u_0) - p_t(u_0) \Delta x_k - \Delta x \left\{ (x_0 g_k u_0) - u_0 K(u_0 - u_1) \right\} \right) = (f_k - x_k C_k) e_k + \left( f_k + 2(x_0 C_k e_k) \right)_u,
\]
where $u_0$ remains to be specified. This gives one equation for each phase.

We next need to derive another equation and a way to get it as follows. As for the momentum equation we would like to write
\[
\frac{\partial}{\partial t} \left( \rho_1 x_1 \right) + \frac{\partial}{\partial x} \left( \rho_1 x_1 \right) = 0
\]
and then, in compliance with (12), (13) and (15),
\[
\sigma_1 \frac{\partial}{\partial t} \left( \rho_1 x_1 \right) + \frac{\partial}{\partial x} \left( \rho_1 x_1 \right) = 2 \left( \sigma_0 C_k e_k \right)_u,
\]
where $\sigma_0$ is of course possible. In practice, we first choose
\[
\sigma_1 = 1 - \chi \left( x_2 \rho_2 T_2 \right),
\]
with
\[
\tilde{\mu} = (1 - \chi) \left( x_2 \rho_2 T_2 \right).
\]

5. The numerical scheme

Let us now give an explicit form of the numerical scheme of solution of the system (1).

5.1. The first step: $U^0 \rightarrow U^{*1/2}$

We first derive the explicit form of the Godunov-type scheme (43) associated with the Riemann solver introduced in the previous section. Using (42), it is a simple matter to check that the numerical flux $G_\alpha(U_1, U_2)$ is given here by
\[
G_\alpha(U_1, U_2) = \begin{pmatrix}
-\frac{1}{2} |u_j'\Delta x_l| \\ \frac{1}{2} F(U_2) + \frac{1}{2} F(U_1) - |\sigma_r| (|U_2'\Delta x_l| - |U_1'\Delta x_l|) \\
-|u_j'(U_2'\Delta x_l) - |\sigma_r| (|U_2'\Delta x_l| - |U_1'\Delta x_l|)
\end{pmatrix}
\]
where
\[
\begin{pmatrix}
G_{\alpha\alpha}(U_1, U_2) \\
G_{\alpha\beta}(U_1, U_2)
\end{pmatrix}
\]
Recall that $u_j' = u_j(U_1, U_2), \ p_i = p_i(U_1, U_2), \ K(u_0 - u_1)(U_1, U_2)$ and $u_0 = u_0(U_1, U_2)$ are consistent approximations of $u_0, p_i, K(u_0 - u_1)$ and $u_0$ respectively. In all the sequel, we will set
\[
\begin{pmatrix}
(u_{\alpha\alpha})^{n+1/2}_i = u_j'(U_j^*, U_j^*), \ (p_{\alpha\alpha})^{n+1/2}_i = p_i(U_j^*, U_j^*), \\
K(u_0 - u_1)_{i+1/2} = (u_0 (U_j^*, U_j^*), \ K(u_0 - u_1)_{i+1/2} = (u_0 (U_j^*, U_j^*))
\end{pmatrix}
\]
Then, replacing $U^{*1}$ by $U^{*1/2}$, the first component of (43) reads here
\[
\begin{pmatrix}
0 = (2x_1)^n - 2 (\chi_1)^n + \Delta t \Delta x \left\{ \left( x_0 g_k u_0 \right) - u_0 K(u_0 - u_1) \right\} \\
0 = \left( (\chi_1)^n - (\chi_1)^n \right) + \Delta t \Delta x \left\{ (\chi_1)^n - (\chi_1)^n \right\}
\end{pmatrix}
\]
or
\[
\begin{pmatrix}
(\chi_1)^n_{i+1/2} = (\chi_1)^n, \ (\chi_1)^n_{i+1/2} = \Delta t \Delta x \left\{ (\chi_1)^n_{i+1/2} - (\chi_1)^n_{i+1/2} \right\}
\end{pmatrix}
\]
with
\[
\phi^+ = \max(\phi, 0), \ \phi^- = \min(\phi, 0).
\]
We thus obtain a natural upwind discretization of the first Eq. (25).

Let us next derive the discrete analogues of the phasic mass, momentum and energy conservation equations. We set
\[
\begin{pmatrix}
m_i(U_1, U_2) \\
\end{pmatrix}
\]
with
\[
\begin{pmatrix}
G_{\alpha\alpha}(U_1, U_2)
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
|u_j'(u_0 g_k u_0 - x_k g_k) - \sigma_r \Delta x_k (u_0 g_k - x_k g_k)
\end{pmatrix}
\]
using the notations
\[
\phi^+ = \max(\phi, 0), \ \phi^- = \min(\phi, 0).
\]
\[
\begin{align*}
\left\{ \begin{array}{l}
q_{ik}(U_i, U_k) = \frac{1}{2} \left( (\sigma_k q_i u_i^2)_{ik} \right. \\
+ (\sigma_k q_i u_i u_k u_i u_k)_{ik} - (\sigma_k q_k u_i u_k)_{ik} \\
- \left. \sigma_k \frac{\partial (u_i u_k)_{ik}}{\partial x} \right) \\
\end{array} \right. \\
\left[ \begin{array}{l}
\left( \sigma_k \frac{\partial (u_i u_k)_{ik}}{\partial x} \right)_{ik} \\
\end{array} \right] = \frac{1}{2} \left( (\sigma_k q_i u_i u_k)_{ik} - (\sigma_k q_k u_i u_k)_{ik} - \sigma_k \frac{\partial (u_i u_k)_{ik}}{\partial x} \right).
\end{align*}
\]

Then the discrete mass conservation equation for the phase \( k \) can be written
\[
\begin{align}
(\sigma_k q_i)_{i+1/2}^{n+1/2} &= (\sigma_k q_i)_{i+1/2}^n - \Delta t \left( \frac{\partial m_i}{\partial x} \right)_i^n.
\end{align}
\]
Next, setting in addition
\[
\begin{align}
\left( \sigma_k \frac{\partial (u_i u_k)_{ik}}{\partial x} \right)_{i+1/2}^{n+1/2} &= \left( \sigma_k \frac{\partial (u_i u_k)_{ik}}{\partial x} \right)_{i+1/2}^n + \Delta t \left( \frac{\partial (u_i u_k)_{ik}}{\partial x} \right)_{i+1/2}^n
\end{align}
\]
the discrete energy conservation equation takes the form
\[
\begin{align}
\left( \sigma_k q_i u_i u_k \right)_{i+1/2}^{n+1/2} &= \left( \sigma_k q_i u_i u_k \right)_{i+1/2}^n - \Delta t \left( \frac{\partial E_i}{\partial x} \right)_i^n
\end{align}
\]
5.2. The second step: \( U^{n+1} \rightarrow U^{n+2} \)

In order to define the solution \( U^{n+1} = (U_i^{n+1}) \), at time \( t^{n+1} \), we propose to simply calculate an approximate value at time \( \Delta t \) of the vector \( U \) obtained from (32). To do so, each integral is approximated using a rectangle formula, leading to
\[
\begin{align}
\left( \sum_{k=1}^{n+1} \right) \frac{u_i}{r_{ik}} (u_i^{n+1/2} - u_i^{n+1/2}) + \Delta t \left( \frac{\partial E_i}{\partial x} \right)_i^n
\end{align}
\]
Together with (73), this yields
\[
\frac{1}{Q_{ll}} = \frac{1}{Q_{ll}} \left( 1 + \frac{j_{ll} + \lambda_{ll}(u_{l} - u_{ll})}{C \lambda_{ll} - j_{l}} \right) = \frac{1}{Q_{ll}} \left( 1 + O \left( \frac{1}{C} \right) \right),
\]
which proves the positivity of $Q_{ll}$ and $Q_{ll}$ again for $C$ large enough.

We can also state obvious properties satisfied by the numerical scheme. First, the void fractions $(a_{k})^{n}$ necessarily remain in the expected interval $[0, 1]$ since they are simply transported by the approximate Riemann solver. Then, the stationary coupling waves between the two phases, that is the ones associated with velocity

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Comparison between exact and numerical solutions of Test 1a at time $t = 3$ and for a 1000-point mesh. From the top to the bottom right: $x$ versus $a_{1}, \rho_{1}, \rho_{2}, u_{1}, u_{2}, \rho_{1}, \rho_{2}$.}
\end{figure}
\(u_i\) are natural solutions of the approximate Riemann solver when the sources are neglected. Thus, they are exactly captured by the numerical scheme. Of course, non stationary coupling waves are also natural solutions of the approximate Riemann solver but as usual are not exactly captured by the scheme due to the averaging procedures inherent to the method. At last, let us mention that the numerical scheme is by construction conservative on the partial mass \(a_kq_k, k = 1, 2\), on the total momentum \(a_1\rho_1u_1 + a_2\rho_2u_2\) and on the total energy \(a_1\rho_1E_1 + a_2\rho_2E_2\).

It would be also interesting to prove the positivity of the specific internal energies \((\varepsilon_k)^n\), and the validity of a discrete entropy inequality related to (17). These issues remain open problems at

---

Fig. 2. Comparison between exact and numerical solutions of Test 1a at time \(t = 3\) and for several mesh sizes. From the top to the bottom right: \(x\) versus \(a_1, \rho_1, \rho_2, u_1, p_1, p_2\).
the moment. Let us mention however that similarly to the continuous level, we suspect that a relevant definition of \( \tilde{p}_I \) (here given by (78) and (79)) could yield the expected entropy inequality.

The asymptotic-preserving property of the scheme which has motivated the whole numerical strategy and in particular the upwind treatment of the sources including them in the definition of the approximate Riemann solver (see again [10] for a very similar approach which turned out to be successful in a simplified context), has not been proved theoretically. However, we will observe in the next section devoted to the numerical experiments that we

---

**Fig. 3.** Comparison between exact and numerical solutions of Test 1b at time \( t = 0.1 \) and for a 300-point mesh. From the top to the bottom right: \( x \) versus \( \rho_1, \rho_2, u_1, u_2, p_1, p_2 \).
can actually recover a pretty good approximation of the asymptotic drift velocity (24).

At last, an extension to deal with the resonance phenomena that emerges in the cases of coalescence of eigenvalues necessitates an additional study.

7. Numerical results

We propose in this section various test cases in order to assess the proposed algorithm. We begin by considering Riemann problems, for which exact solutions are available, and compare the

Fig. 4. Comparison between exact and numerical solutions of Test 1b at time \( t = 0.1 \) and for several mesh sizes. From the top to the bottom right: \( x \) versus \( \rho_1, \rho_2, u_1, u_2, \rho_1, \rho_2 \).
numerical results with the ones provided by alternate methods available in the literature. More precisely, we will consider the HLL-type method proposed by Saurel and Abgrall [37], the strategy presented by Andrianov et al. [4], the VFRoe scheme derived by Gallouët et al. [20], and at last the Godunov scheme of Schwendeman et al. [40]. Note from now on that the comparison will be based on simulations and pictures proposed in these papers only.

We will then go on considering more typical two-phase flow simulations, namely the Ransom water faucet problem, the sedimentation test case, a desequilibrium in velocities test case, and

Fig. 5. Comparison between exact and numerical solutions of Test 2 at time $t = 0.2$ and for several mesh sizes. From the top to the bottom right: $x$ versus $\alpha_1$, $\rho_1$, $\rho_2$, $u_1$, $u_2$, $\rho_1$, $\rho_2$. 
at last a vertical bubbly column. These simulations involve non zero source terms which allows to assess our numerical upwind strategy for the external and drag forces. In particular, in the last test case, a specific attention will be paid to the (numerical) asymptotic preserving property of the scheme.

Except otherwise stated, all our simulations are performed on the interval [0, 1] and the (possible) Riemann initial discontinuity is always located at point $x = 0.5$. Our numerical scheme will be referred to as Godunov-type scheme in the following pictures.

### 7.1. Riemann problems

We consider in this part a Riemann initial data

$$U(x, 0) = U_0(x) = \begin{cases} U_1 & \text{if } x < 0, \\ U_R & \text{if } x > 0, \end{cases}$$

where for the sake of clarity, the initial states $U_i$ and $U_R$ will be defined from the following set of initial values in primitive variables: $\alpha_1, \rho_1, u_1$ and $\rho_2, p_2$, $k = 1, 2$. Regarding the closure laws, we take just for simplicity the classical ideal gas equation of state

$$p_k = (\gamma_k - 1) \rho_k e_k, \quad k = 1, 2,$$

with $\gamma = \gamma_1 = \gamma_2 = 1.4$.

### 7.1.1. Test 1: isolated coupling wave

In the first two simulations, we consider the simple situation of an isolated coupling wave propagating with velocity $u_c$. The first simulation (Test 1a) is taken from [20] and corresponds to the choice $\chi = 0.5$, that is

$$U_i = \frac{\alpha_1 \rho_1 u_1 + \alpha_2 \rho_2 u_2}{\alpha_1 \rho_1 + \alpha_2 \rho_2}.$$

We choose $\alpha_{1, L} = 0.9$, $(\rho_1, u_1, p_1)_L = (1.100, 10^5)$, $(\rho_2, u_2, p_2)_L = (1.100, 10^3)$, $\alpha_{1, R} = 0.5$, $(\rho_1, u_1, p_1)_R = (0.125, 100, 10^3)$, $(\rho_2, u_2, p_2)_R = (0.125, 100, 10^3)$.

Solutions are presented in Fig. 1 and the results are compared with the ones given by the VFRoe scheme [20]. We note that both schemes preserve the constant velocities and pressures by construction, and present exactly the same numerical diffusion. Fig. 2 shows the solutions provided by our scheme on several meshes.

The second simulation (Test 1b) still corresponds to an isolated coupling wave, but now associated with the choice $\chi = 1$, leading to $u_c = u_1$. Initial conditions are taken from [5] and given by $\alpha_{1, L} = 0.8$, $(\rho_1, u_1, p_1)_L = (2.0, 3.5)$, $(\rho_2, u_2, p_2)_L = (1.2, 1)$, $\alpha_{1, R} = 0.3$, $(\rho_1, u_1, p_1)_R = (2.0, 3, 12.8567)$, $(\rho_2, u_2, p_2)_R = (0.1941, 2.8011, 0.1)$.

We note that the pressures and velocity $u_2$ are not taken to be equal anymore. Solutions are presented on Fig. 3. We observe that the results are in good agreement with the exact solutions. They also compare very well with the ones given in [5]. The right state of the coupling wave, in particular for the $u_2$ component, agrees well with the exact value. The small amplitude oscillations observed on $\rho_1$ and $u_1$ are classical and due to the initial pressures and velocity $u_2$ disequilibrium. Fig. 4 shows the numerical solutions obtained with several mesh sizes.

### 7.1.2. Test 2: a general Riemann problem

As a general Riemann problem, we consider here a test case taken from [40] for which the exact solution is known (see Table 1, p. 82).
Fig. 8. Behaviour of $a_1(x)$, $p_1(x)$, $q_1(x)$, $q_2(x)$ at time $t = 0.2$ and for Test 2. These pictures are taken from [40] by courtesy of the authors. The curves are given by method $G_{HL}$ (blue), $G_{ASH}$ (green) and $G_1$ (red). The exact solution appears in black. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
and comparisons with other methods from the literature are available.

Initial conditions are such that the mixture is at rest and presents a (negative) jump in the volume fraction $\alpha_1$. The density $\rho_1$ and pressure $p_1$ are equal on each side of the initial discontinuity. More precisely, the initial states $U_{i,k}$ are calculated from the following values in primitive variables:

![Diagram showing comparisons for different values of $\chi$.](image-url)

Fig. 9. Comparison between numerical solutions of Test 3 at time $t = 0.2$ and for several values of $\chi$. The mesh is made of 400 points. From the top to the bottom right: $x$ versus $\alpha_1, \rho_1, \rho_2, u_1, u_2, p_1, p_2$. 

Fig. 10. Comparison between numerical solutions of Test 4 at time $t = 0.5$ given by the VFRoe scheme and the Godunov-type scheme and for several mesh sizes. On the left: $\varphi_2$, on the right: $u_1$. From the top to the bottom: 200-point mesh, 2000-point mesh and 20,000-point mesh.

Fig. 11. Comparison between numerical solutions of Test 4 at time $t = 0.5$ given by the Godunov-type scheme and for several mesh sizes. On the left: $\varphi_2$, on the right: $u_1$. 

\( \alpha_{1L} = 0.8 \), \( (\rho_1, u_1, p_1)_L = (1, 0, 1) \)
\( (\rho_2, u_2, p_2)_L = (0.2, 0.0, 0.3) \),
\( \alpha_{1R} = 0.3 \), \( (\rho_1, u_1, p_1)_R = (1, 0, 1) \)
\( (\rho_2, u_2, p_2)_R = (1, 0, 1) \).

leading to an exact solution made of a combination of shock, contact discontinuity and rarefaction waves, in addition to the expected coupling wave associated with \( u_i = u_1 \). All the solutions are plotted at time \( t = 0.2 \).

We first give on Fig. 5 the results obtained by our algorithm with a mesh composed of 200, 400 and 800 cells and compare the numerical and exact solutions. We observe that the phase 2 presents a good agreement while the phase 1 suffers from over-shoots and undershoots at the extremes waves. However and as it will be seen below, most previously proposed schemes exhibit the same behavior. In addition, we note the void fraction, and more generally the coupling wave, is very well computed in the sense that the left and right states associated with \( u_i \)-discontinity agree with the exact values. In order to illustrate this good property, we proceed as in [40] and compare in Fig. 6 the behavior of the numerical solutions close to the coupling wave with that determined by the exact jump conditions across this wave. These are provided by the five \( u_i \)-Riemann invariants, given in this case by (see again [40]):

\[
\begin{align*}
K_0 &= u_1, \\
K_1 &= (1 - \alpha_1) \rho_2 (u_2 - u_1), \\
K_2 &= (1 - \alpha_1) \rho_2 (u_2 - u_1)^2 + \alpha_1 p_1 + (1 - \alpha_1) p_2, \\
K_3 &= \frac{\rho_0}{\rho_1} - \frac{1}{2} (u_2 - u_1)^2, \\
K_4 &= \frac{p_0}{\rho_2}.
\end{align*}
\]

For \( \alpha_1 \) varying from \( \alpha_{1L} \) to \( \alpha_{1R} \), these equations provide a parametrization of the exact (vanishingly thin) layer of the \( u_i \) coupling wave. This one is compared with the numerical layer induced by the numerical diffusion of the scheme. Note that the five constants \( K_i, i = 0, \ldots, 4 \) are calculated using the exact value of the right state corresponding to \( \alpha_1 = \alpha_{1R} \) as given in [40]. As expected, Fig. 6 shows a good agreement between exact and numerical layers.

We now compare our numerical solutions in Fig. 7 with the ones proposed by Saurel and Abgrall [37] (referred to as GHLL), by Andrianov et al. [4] (referred to as G_{SAW}) and by Schwendeman et al. [40] (referred to as G_{S}) on Fig. 8. Notice that the latter one corresponds to an exact Godunov method (i.e. based on the exact resolution of the Riemann problem) and is thus expected to be the most precise. This is actually the case, we indeed observe that with this method, all the intermediate states are perfectly computed and that the solutions do not present any odd behavior next to the acoustic waves. As far as the other methods are concerned, we first observe that the GHLL method proposed by Saurel and Abgrall [37] is the most diffusive on this test case. The G_{SAW} method of Andrianov et al. [4] is less diffusive but does not provide us with a very good approximation of the constant states of the \( u_i \) coupling wave. On the contrary, we can observe that our approach behaves very well in the vicinity of this coupling wave (while still presenting overshoots and undershoots next to the extreme waves of the solution). In particular, the left and right states of the \( u_i \)-contact wave appears to be better evaluated.

7.1.3. Test 3: a general Riemann problem with several values of \( \chi \)

To conclude this first batch of test cases, let us emphasize that our algorithm allows by construction to deal with several values of the interfacial velocity \( u_i \) and pressure \( p_i \), depending on the underlying modelling assumptions. In particular, we have seen that admissible values for the couple \( (u_i, p_i) \) may be parametrized by the parameter \( \chi \in [0, 1] \). We propose now to take advantage of this flexibility of our algorithm to compare the solutions obtained for several values of \( \chi \). The Riemann initial data is the same as for Test 2. So far, we dealt with \( \chi = 1 \) for the sake of comparison with the results presented in [37,40]. We now also consider the natural values \( \chi = 0 \) and \( \chi = 0.5 \). The results are presented in Fig. 9. We observe in particular that the values \( \chi = 0 \) and \( \chi = 0.5 \) now present an even behavior in the vicinity of the acoustic waves (unlike the value \( \chi = 1 \)). Above all, we note that the three solutions are actually different, highlighting the importance of having a scheme capable of dealing with various values of \( \chi \).

Remark. We have not been able to pass the latter test case with the VFRoe scheme proposed in [20], probably due to the lack of positivity properties of the VFRoe approach. On the contrary, let us mention that unlike our approach the VFRoe scheme turns out to be able to compute solutions involving the resonance phenomenon, see Test 2 in [20]. Recall indeed that we have assumed here from the beginning that the waves are ordered such that the acoustic waves propagate faster. More generally, if we consider Riemann initial data such that both our approach and the VFRoe scheme succeed in computing an approximate solution, we observed a good agreement between them. It is worth noticing however that the numerical diffusion turned out to be equivalent for both scheme across the coupling wave (as observed in Test 1a above) and the acoustic waves, but a little less important for the VFRoe scheme across the contact discontinuities associated with \( u_1 \) and \( u_2 \). This is clearly expected since these contact discontinuities are not present in our approximate Riemann solver. These numerical results are not reported here.
7.2. Influence of the source terms

Let us now take into account the source terms effects in our simulations. We begin this section with the well-known Ransom faucet and sedimentation problems where only the external forces (namely the gravity) and the relaxation pressure term in the void fraction equation are accounted for. The next simulation considers only the interfacial drag force and the relaxation pressure term. The fourth one involves both the external and drag forces and the relaxation pressure term, and focus on the asymptotic behavior of the model under consideration when the underlying relaxation times are small.

Fig. 14. Stationary solution of Test 6. From the top to the bottom right: $x$ versus $x_1$, $p_1$, $p_2$, $u_1$, $u_2$, $p_1$, $p_2$. 
We consider again just for simplicity the classical ideal gas equation of state

\[ p_k = (\gamma_k - 1)\rho_k v_k, \quad k = 1, 2, \]

but now with \( \gamma_1 = 1.00005 \) and \( \gamma_2 = 1.4. \)

In order to define the interfacial velocity and pressure, we take \( \chi = 0.5. \)

As far as the relaxation pressure term is concerned, we take

\[ K_p(U) = \frac{1}{\tau_p} \left( \frac{1 - \alpha}{\rho_1} + \frac{\alpha}{\rho_2} \right) (p_1 - p_2), \]

where the relaxation time \( \tau_p \) has to be specified.

At last, the forthcoming test cases involves boundary conditions at the end points \( x_{\min} \) and \( x_{\max} \) of the domain (\( x_{\min} = 0 \) and \( x_{\max} = 1 \) except otherwise stated). Regarding the numerical treatment of these boundary conditions, we have used two classical fictitious states we call here \( U_{\min} \) and \( U_{\max}. \) They will be made precise for each test case.

7.2.1. Test 4: the Ransom water faucet problem

This test case is a very classical benchmark test [35]. The configuration corresponds to a 12 m long vertical tube (\( x_{\min} = 0 \) and \( x_{\max} = 12 \)). The initial condition consists of a uniform column of water (indexed by 1) in the air (indexed by 2). The void fraction is such that \( \alpha_1 = 0.8. \) The velocities \( u_1 \) and \( u_2 \) are also uniform and respectively equal to 10 m/s and 0 m/s. All the pressures are set to be equal to \( 10^5 \) Pa. Initial densities are such that \( \rho_1 = 10^3 \) kg/m\(^3\) and \( \rho_2 = 1 \) kg/m\(^3\). Since the constant gravity field \( g = 9.81 \) m/s\(^2\) is present, both phases separate and the solution at time \( t = +\infty \) is made of two regions of pure liquid (at the bottom of the tube, \( x_1(x_{\min} = 0) = 1 \) and gas (at the top of the tube, \( x_1(x_{\max} = 12) = 0 \)). From a numerical point of view, the fictitious states are such that

\[ \alpha_{1,\min} = 0.8, \quad (\rho_1, u_1, p_1)_{\min} = (10^3, 10^3, 0), \quad (\rho_2, u_2, p_2)_{\min} = (0, 0, 10^5) \]

and

\[ p_{1,\max} = p_{2,\max} = 10^5. \]

Classical Neumann numerical boundary conditions are applied to define

\[ \alpha_{1,\max}, \quad (\rho_1, u_1)_{\max}, \quad (\rho_2, u_2)_{\max} \]

(the values are taken from inside the domain). At last, the time scale is \( \tau_p = 5 \times 10^{-4} \) s in the pressure relaxation term.

The numerical results are plotted in Fig. 10 and compared with the ones given by the VF Roe scheme proposed in [20]. Solutions are very similar. In Fig. 11 are superimposed the results obtained with our scheme for various mesh sizes. We note in particular that an usual problem encountered when dealing with a two-fluid one-pressure model is related to the presence of complex wave speeds when the mesh size becomes too small (approximatively more than 1000 cells, see [20]) and the appearance of negative volume fractions. Here, the two-fluid two-pressure model under consideration is always (possibly weakly) hyperbolic. The finest mesh we have considered here is made of 20,000 cells and did not raise any difficulty.

7.2.2. Test 5: the sedimentation test case

Again, this test case is a classical benchmark test for the simulation of two-phase flows [12]. The configuration corresponds to a 7.5 m long vertical tube, that is \( x_{\min} = 0 \) and \( x_{\max} = 7.5. \) The initial condition consists of a uniform mixture of liquid (indexed by 1) and gas (indexed by 2) such that \( \alpha_1 = 0.5. \) The initial velocities \( u_1 \) and \( u_2 \) are equal to 0 m/s, and both pressures are equal with \( p_1 = p_2 = 10^5 \) Pa. Initial densities are such that \( \rho_1 = 10^3 \) kg/m\(^3\) and \( \rho_2 = 1 \) kg/m\(^3\). The domain is closed, meaning that classical rigid wall boundary conditions are imposed at the boundaries: \( u_1(x_{\min} = 0, t) = u_2(x_{\max} = 7.5, t) = 0, \quad k = 1, 2. \) Since the constant gravity field \( g = 9.81 \) m/s\(^2\) is present, both phases separate and the solution at time \( t = +\infty \) is made of two regions of pure liquid (at the bottom of the tube, \( x_1(x_{\min} = 0) = 1 \) and gas (at the top of the tube, \( x_1(x_{\max} = 7.5) = 0 \)). From a numerical point of view and as it is usual, the fictitious states are obtained from classical numerical Neumann boundary conditions applied to the primitive variables, except for the velocities \( u_{l,\min} \) and \( u_{l,\max} \) that are classically given the opposite values of the velocities \( u_l \) coming from inside the domain (\( k = 1, 2 \)).

At last, let us mention that in order to avoid natural numerical instabilities due to phase vanishing, we have modified in our code, at each time iteration, the updated values of both velocities \( u_1 \) and \( u_2 \) and set them to \( u_1 = u_2 = 0 \) as soon as either \( \alpha_1 \) or \( \alpha_2 \) is less than a threshold \( \eta. \) Hereafter, \( \eta = 3 \times 10^{-5}. \)

In Fig. 12, we give the numerical profile of the void fraction \( \alpha_1 \) at different intermediate times and for a mesh made of 400 cells. The time scale is taken to be \( \tau_p = 5 \times 10^{-3} \) s in the pressure relaxation term. We observe that the solutions at times \( t = 1.2 \) and \( t = 1.4 \) are identical so that the stationary profile of \( \alpha_1 \) seems to be reached. We also note that the final void fraction discontinuity is not located at the middle point \( x = 3.75 \) of the domain due to the compressibility effects.

In Fig. 13, we plot the solution obtained with \( \tau_p = 5 \times 10^{-4} \) s and a 4000-cell mesh. The pressure relaxation time is thus faster and the mesh finer (the latter turned out to be necessary to get the numerical stability of the algorithm). We observe that the stationary profile appears to be attained more quickly.
7.2.3. Test 6: velocities desequilibrium

This test case corresponds to the evolution of a two-phase flow in a duct of uniform section and 1 m length. In this case, the external forces are assumed to be negligible but we consider now a non-zero interfacial drag force given by

\[
K(U) = \frac{1}{8} C_D a^{int} \rho_1 |u_1 - u_2|.
\]  
(94)

In this formula, the coefficients

\[
C_D = 0.5, \quad a^{int} = \frac{3(1 - \alpha_1)}{r_b}, \quad r_b = 2.5 \times 10^{-4},
\]  
(95)
represent a drag coefficient, an interfacial area and a mean bubbles radius respectively. Then, index 1 (resp. 2) is sort of a liquid (resp. vapor) phase.

The pressure relaxation time is taken to be \( \tau_p = 0.5 \times 10^{-3} \).

Theoretical boundary conditions are such that at the inlet \( x_{\text{min}} = 0 \) we impose a non zero relative velocity such that \( u_1 = 1 \text{ m/s} \) and \( u_2 = 1.5 \text{ m/s} \). The void fraction \( x_1 = 0.8 \) and the density values \( \rho_1 = 10^3 \text{ kg/m}^3 \) and \( \rho_2 = 1 \text{ kg/m}^3 \). At the outlet \( x_{\text{max}} = 1 \), we only impose the pressures \( p_1 = p_2 = 101,300 \text{ Pa} \).

From a numerical point of view, the fictitious states are such that

\[
x_1_{\text{min}} = 0.8, \quad (\rho_1, u_1)_{\text{min}} = (10^3, 1), \quad (\rho_2, u_2)_{\text{min}} = (1.5)
\]

and

\[
p_1_{\text{max}} = p_2_{\text{max}} = 101,300.
\]

Classical Neumann numerical boundary conditions are applied to define the others quantities.

Initial conditions are taken to be uniform in the computational domain and such that \( u_1 = u_2 = 0 \), the others variable been chosen in agreement with the boundary conditions.

Our objective here is to test the ability of the method to compute the stationary solution associated with such a stiff source terms problem. First, due to the interfacial drag force, the relative velocity \( u_1 = u_2 - u_1 \) is expected to tend to zero very fast along the pipe (see [31]). Then, due to the relaxation pressure term, the relative pressure \( p_1 = p_2 - p_1 \) is expected to tend to zero as well. From the void fraction evolution equation, \( x_1 \) is thus expected to become constant very fast along the pipe, as well as the pressures \( p_1 \) and \( p_2 \) (recall that the gravity field is not present here). From the mass and momentum equations, we easily see that the same happens for the densities and the partial velocities.

We plot in Figs. 14 and 15 the stationary profiles given by a 2000-point mesh. We can observe that they are in good agreement with the expected behavior.

7.2.4. Test 7: vertical bubbly column

This test case corresponds to the evolution of a two-phase flow in a vertical column of 1 m length. In this case, both the gravity \( g = -9.81 \text{ m/s}^2 \) and a non zero interfacial drag force are considered. The latter is given by the same formula as in the previous test case. In the pressure relaxation source term, the relaxation time is taken to be \( \tau_p = 1 \times 10^{-4} \).

Theoretical boundary conditions are such that at the inlet \( x_{\text{min}} = 0 \), we impose a non zero relative velocity such that \( u_1 = 5 \text{ m/s} \) and \( u_2 = 15 \text{ m/s} \), the void fraction \( x_1 = 0.97 \), the density values \( \rho_1 = 10^3 \text{ kg/m}^3 \) and \( \rho_2 = 1 \text{ kg/m}^3 \), and the mixture pressure \( p = 155 \times 10^5 \text{ Pa} \). At the outlet \( x_{\text{max}} = 1 \), we only impose the mixture pressure \( p = 150 \times 10^5 \text{ Pa} \).

From a numerical point of view, the fictitious states are such that

\[
x_1_{\text{min}} = 0.97, \quad (\rho_1, u_1, p_1)_{\text{min}} = (10^3, 5, 155 \times 10^5), \quad (\rho_2, u_2, p_2)_{\text{min}} = (1, 15, 150 \times 10^5)
\]

and

\[
p_1_{\text{max}} = p_2_{\text{max}} = 150 \times 10^5.
\]

Classical Neumann numerical boundary conditions are applied to define the others quantities.

Initial conditions are taken to be uniform in the computational domain and in agreement with the inlet boundary conditions for all the variables except the pressures that are defined from the outlet boundary values.

We plot in Fig. 16 the stationary profiles given by a 400-point mesh. Our objective here is to assess the capability of our algorithm to restore at the numerical level a specific asymptotic behavior of the solutions of the two-pressure diphasic model under consideration in this paper. More precisely, it is recalled in Section 2.3 that if the time scales of the pressure relaxation and drag force source terms involved in the model are much smaller than the...
other characteristic time scales of the system, and provided that we have a balance of the gradient of pressure with the external forces, which writes here
\[ \partial_t p = -\rho g, \quad p = \rho_1 \rho_1 + \rho_2 \rho_2, \quad \rho = \rho_1 \rho_1 + \rho_2 \rho_2, \]
then the solutions behave like the solutions of a drift-flux model. Such a model is governed by conservation laws on mixture density, mixture momentum, mixture total energy and mass fraction of one of the two phases, and involves a closure law on the relative velocity \( u = u_2 - u_1 \). From (24), (94) and (95) the corresponding drift law is
\[ \frac{1}{8} \mathcal{C}_D \rho_1 |u_1| |u_2| = \frac{1}{2} \left( 1 - \frac{\rho_1}{\rho_2} \right) g. \]

Note that in the present test case, we have \( u_1 < u_2 \) and \( \rho_2 < \rho_1 \). We then easily get \( u > 0 \), so that taking into account the values of the different coefficients,
\[ u^2 = -\frac{1}{750} \left( 1 - \frac{\rho_1}{\rho_2} \right) g \]
(recall that \( g < 0 \)).

We can observe in Fig. 17 that this asymptotic drift law is numerically satisfied, as well as the expected balance between the gradient of pressure and the gravity field.

Acknowledgments

The authors are very grateful to F. Coquel and N. Seguin for the very beneficial discussions we had on this work.

References