Parametrized Asynchronous Shared-Memory Systems

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1. The model
2. (Non-)Deterministic Model Checking
3. Probabilistic case

1 The model
- Definitions
- Example

2 (Non-)Deterministic Model Checking

3 Probabilistic case

Definition (Distributed protocol)

A distributed protocol is given by $\mathcal{P} = \langle Q, D, T \rangle$

- Q: control states
- D: possible values of the register
- T: transitions of the form $p \xrightarrow{r(d)} q$ and $p \xrightarrow{w(d)} q$ for $p, q \in Q$, $d \in D$.

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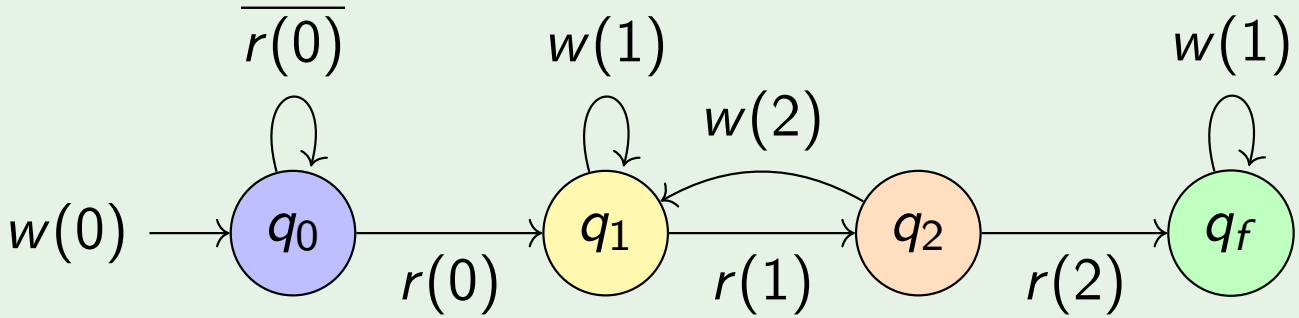
Example



Definition (Configuration of the protocol)

$$\gamma = \langle f, d \rangle$$

with $f : Q \to \mathbb{N}$ (multiset) and $d \in D$ the register value. We write $\gamma(q) = f(q)$ and $v(\gamma) = d$.

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Some notations:

- Γ is the set of configurations
- $|\gamma| = \sum_q \gamma(q)$ (size)
- $\overline{S}(\gamma) = (\{q \mid \gamma(q) > 0\}, v(\gamma))$
- $\mathrm{Pre}(X)$, $\mathrm{Post}(X)$
- $+$, $-$ operations on multisets are extended to configurations.

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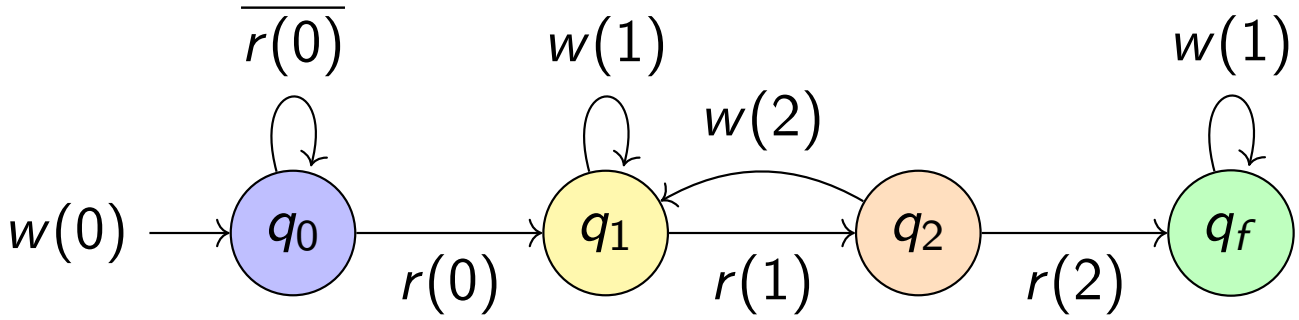
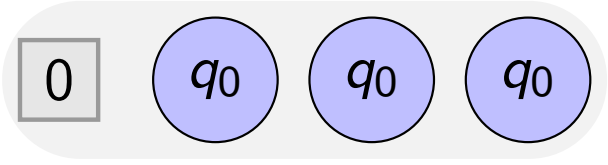
Definition (Semantics)

$\gamma \to \gamma'$ if $\gamma' = \gamma - q + q'$ with either

- $q \xrightarrow{w(v(\gamma))} q'$ (write operation)
- or $d = v(\gamma) = v(\gamma')$ and $q \xrightarrow{r(d)} q'$ (read operation)

Semantics





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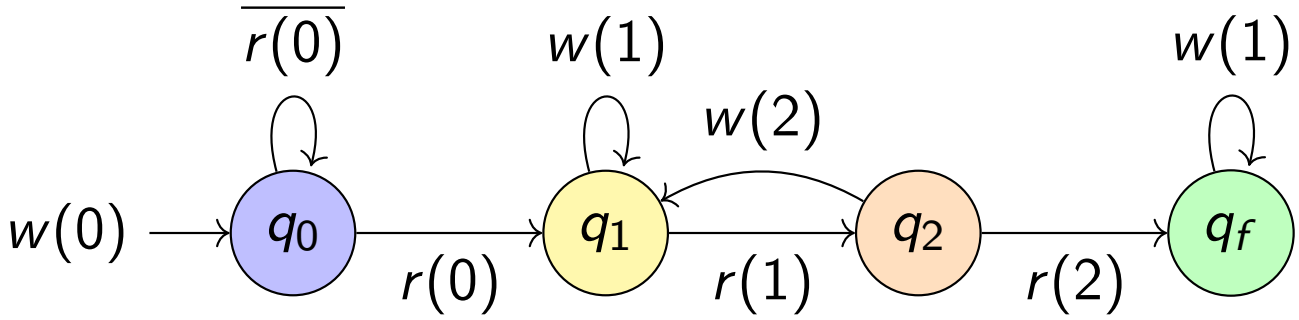

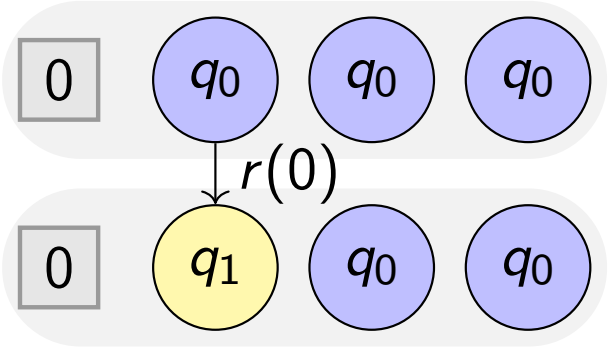

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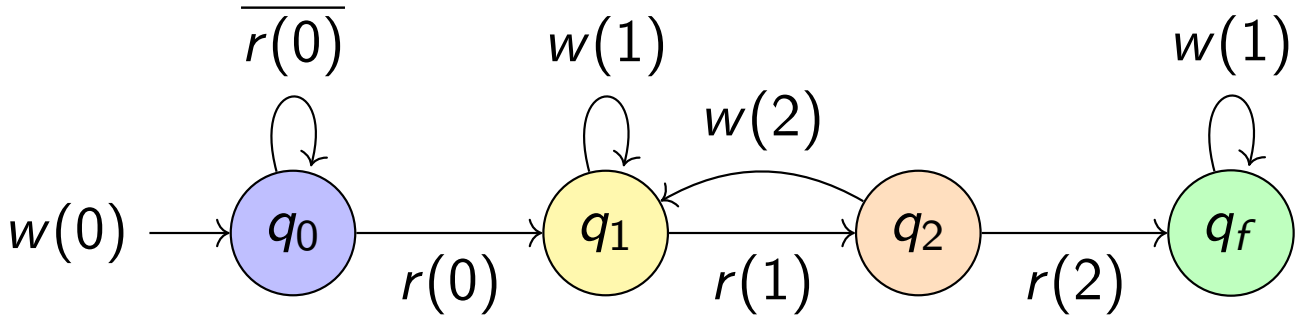
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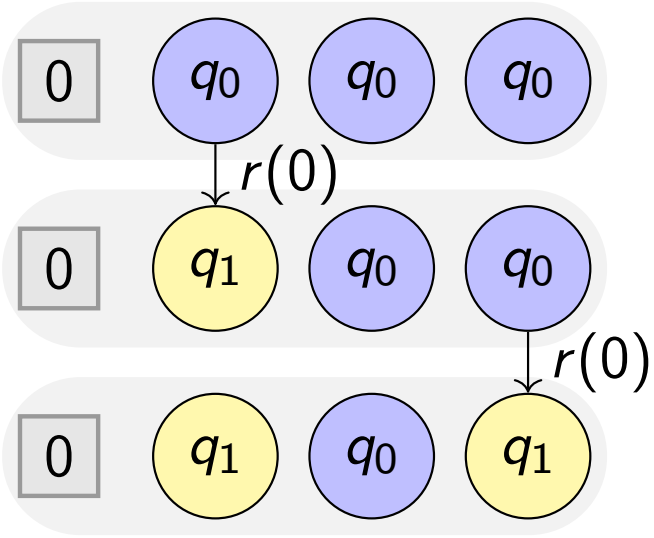
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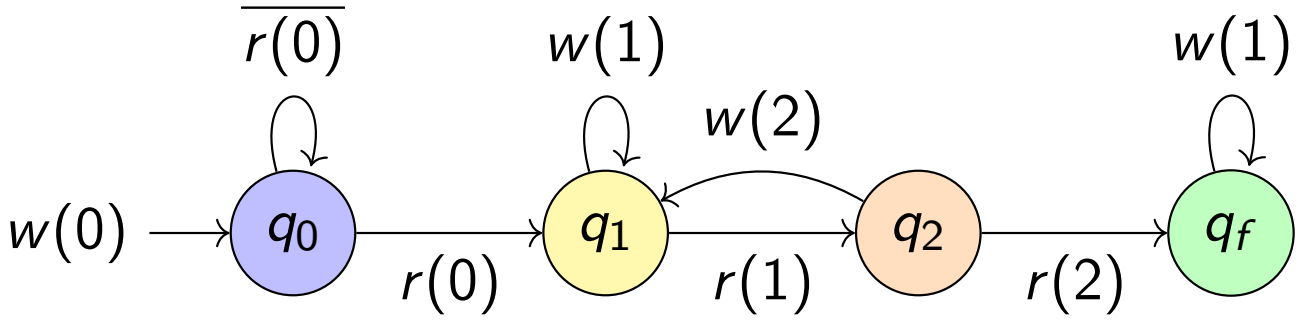
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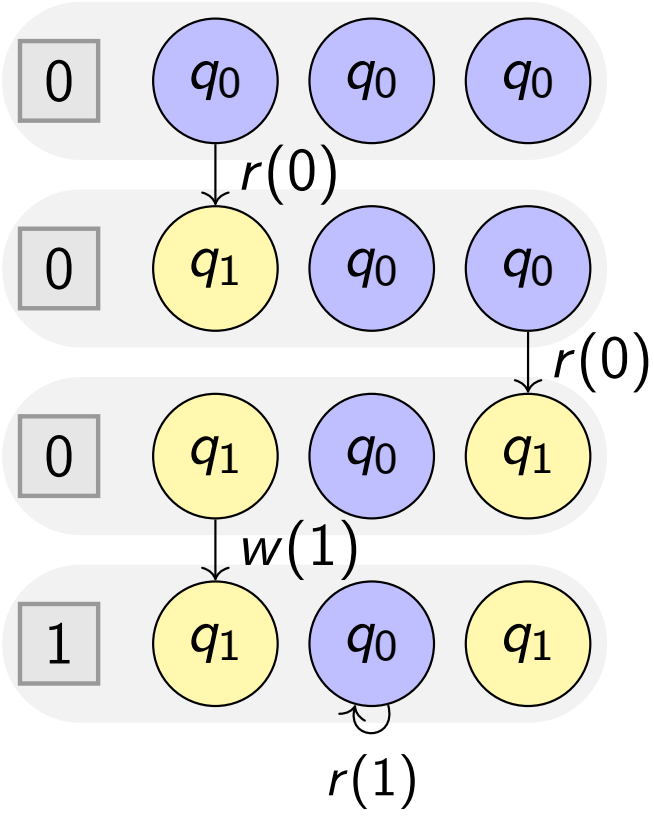
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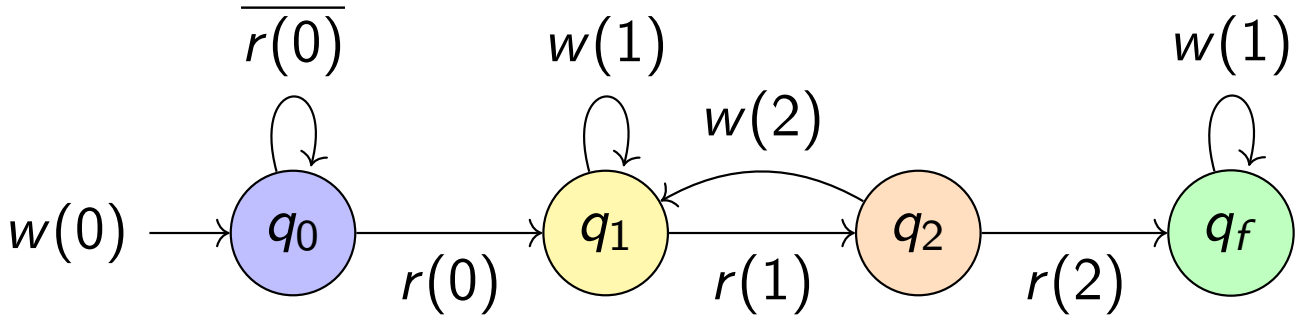
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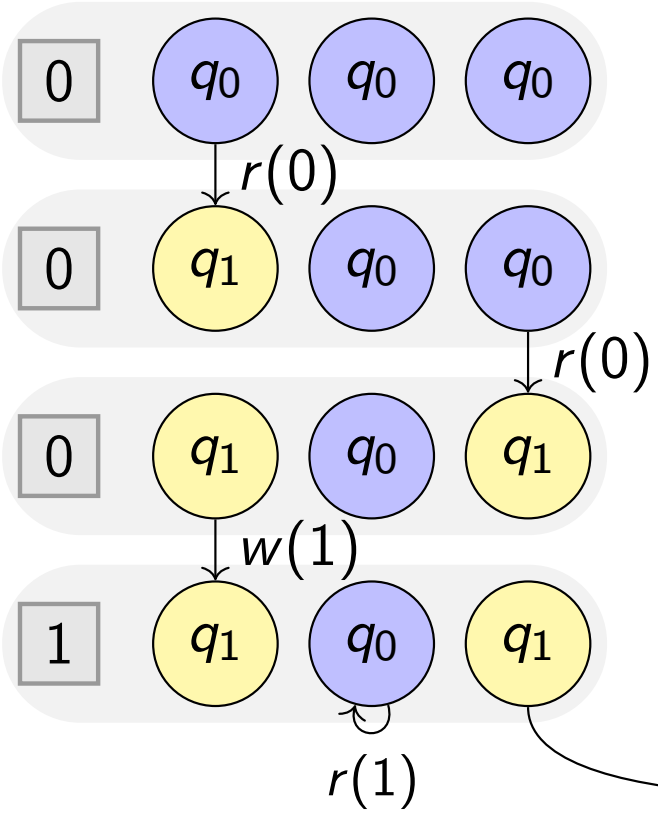
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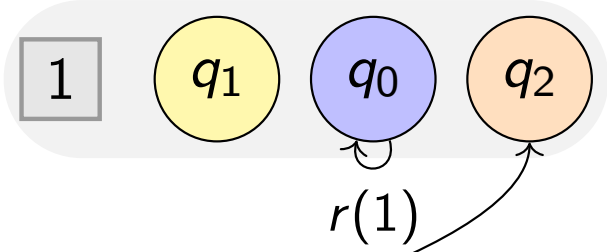
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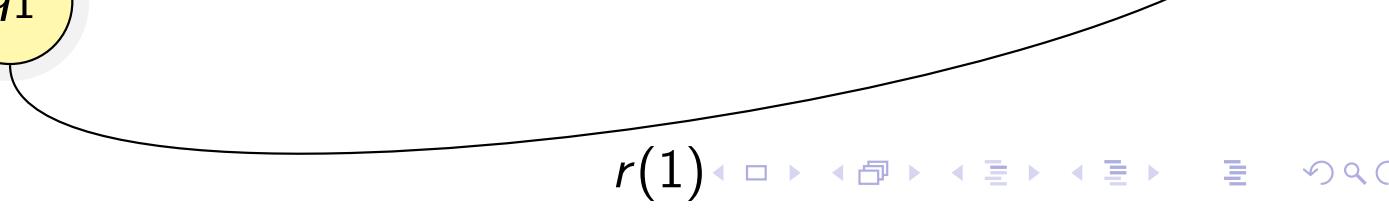
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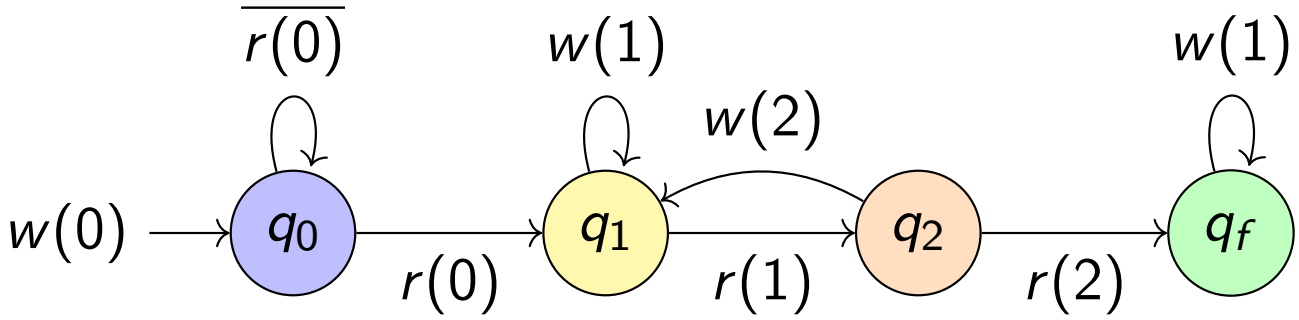
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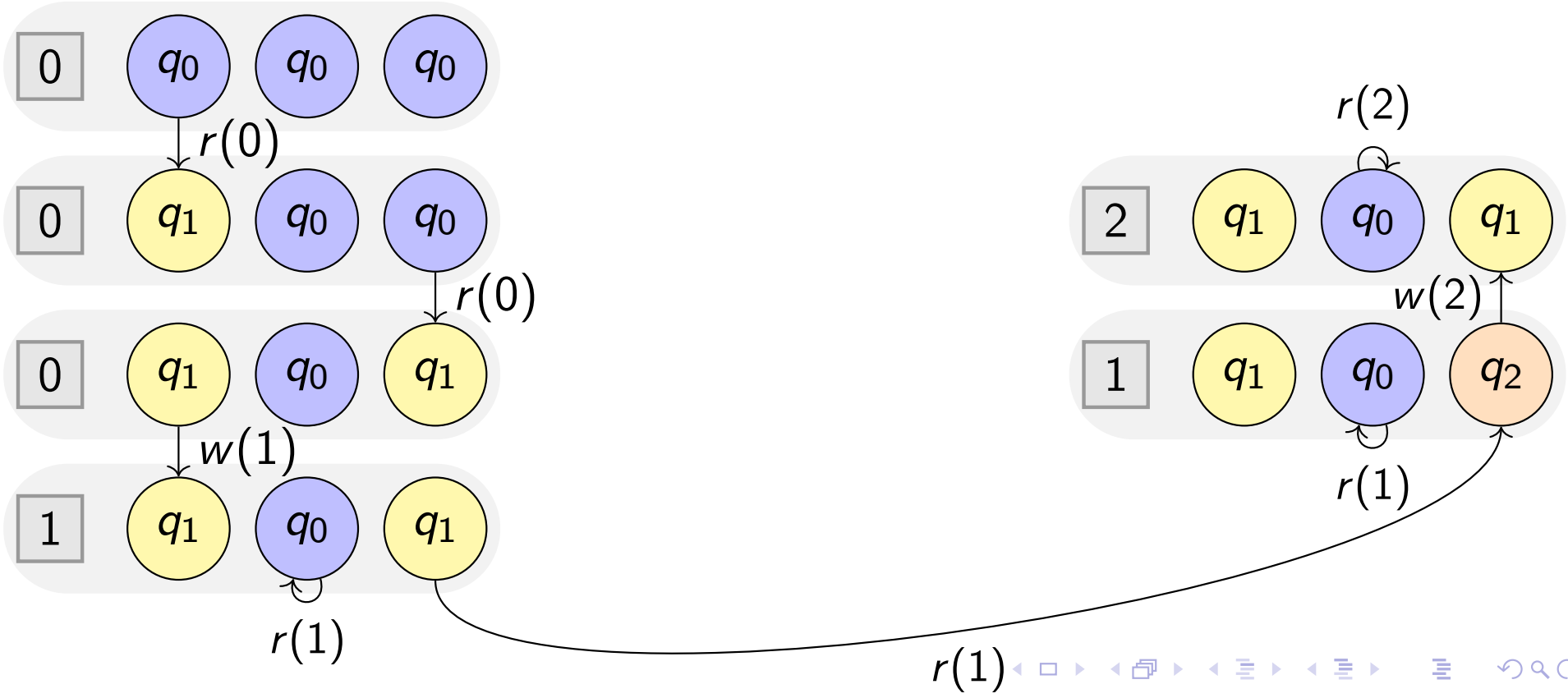
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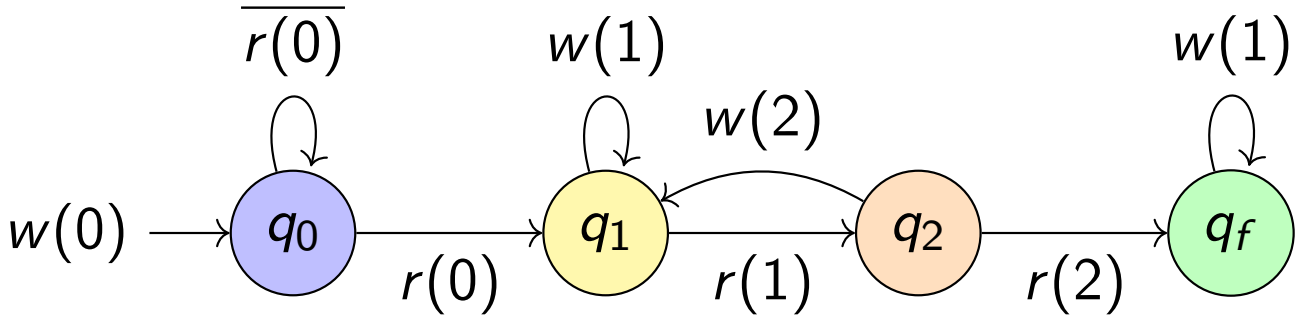
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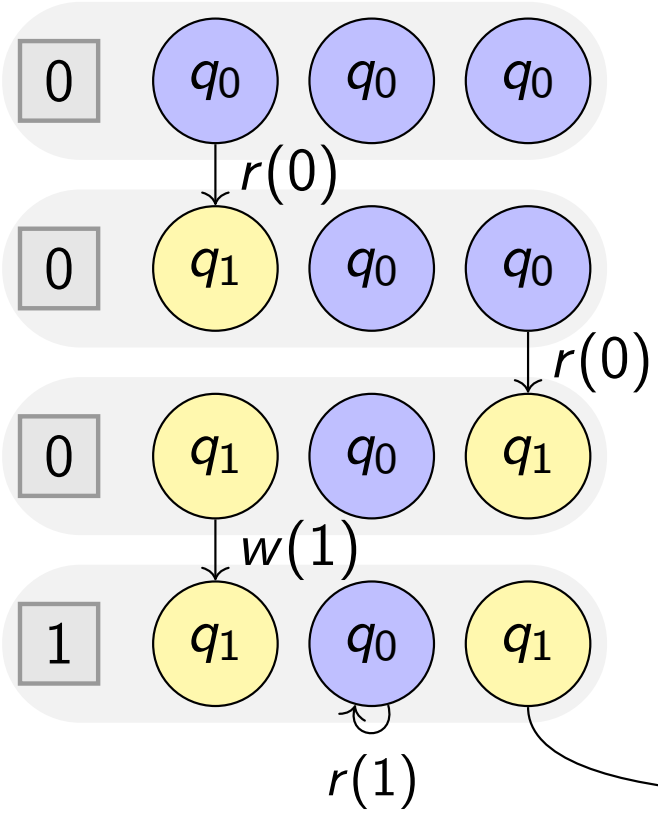
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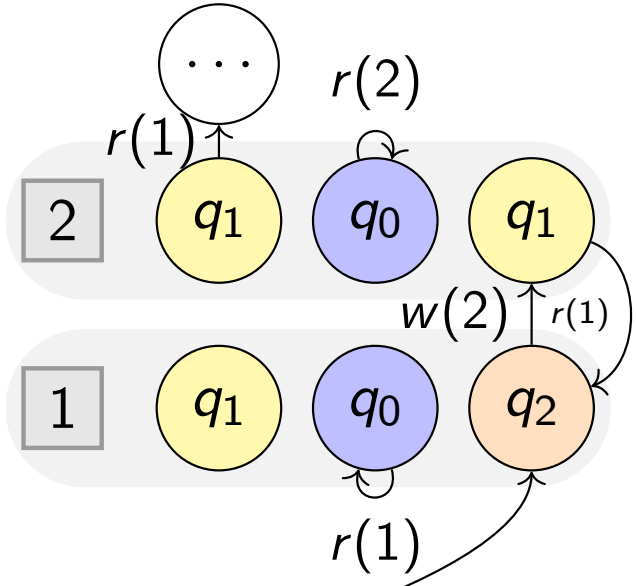
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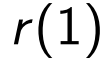
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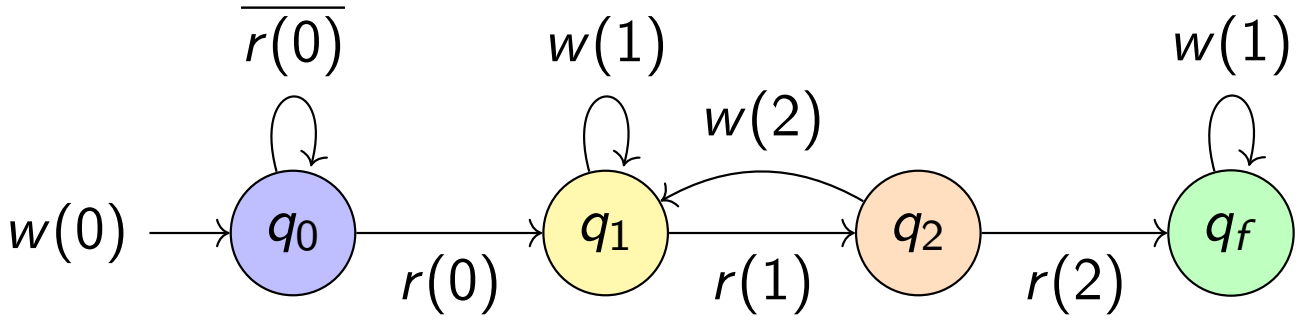
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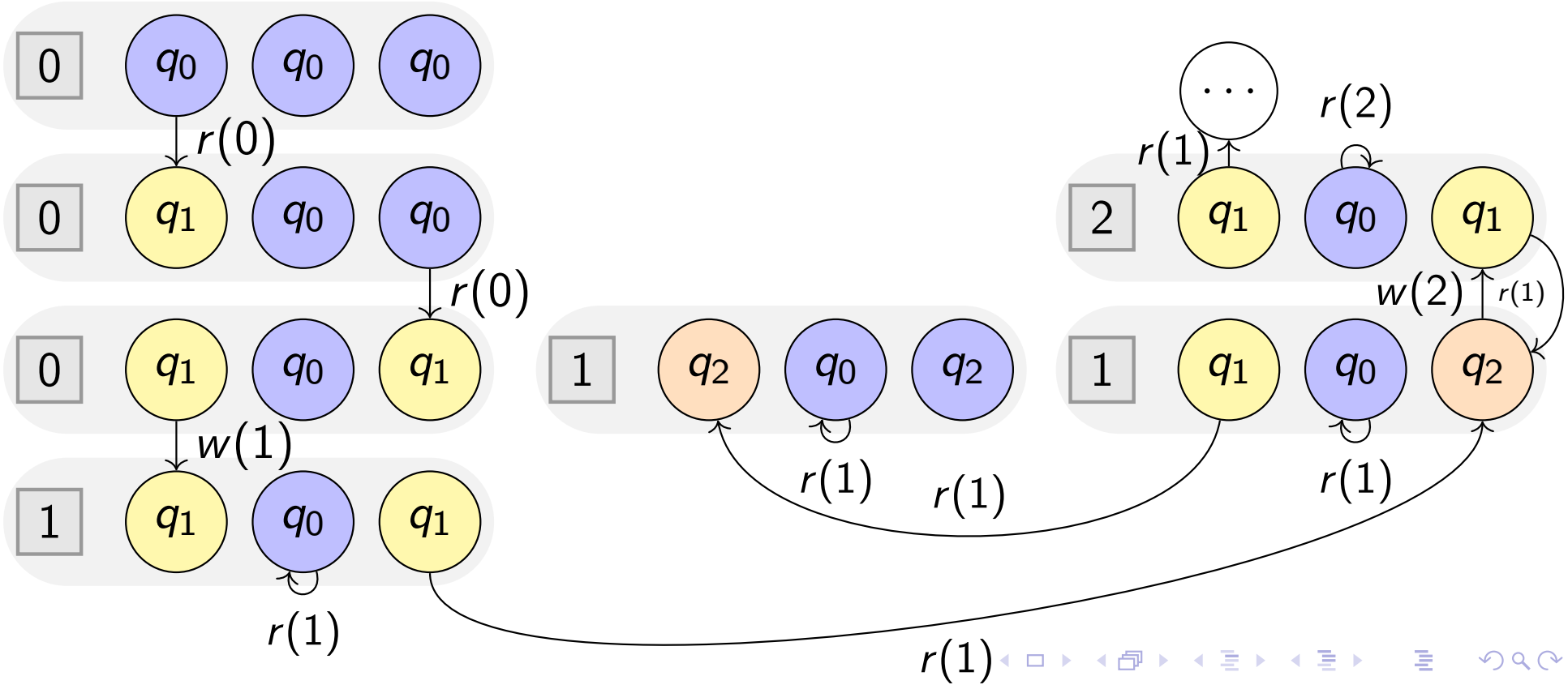
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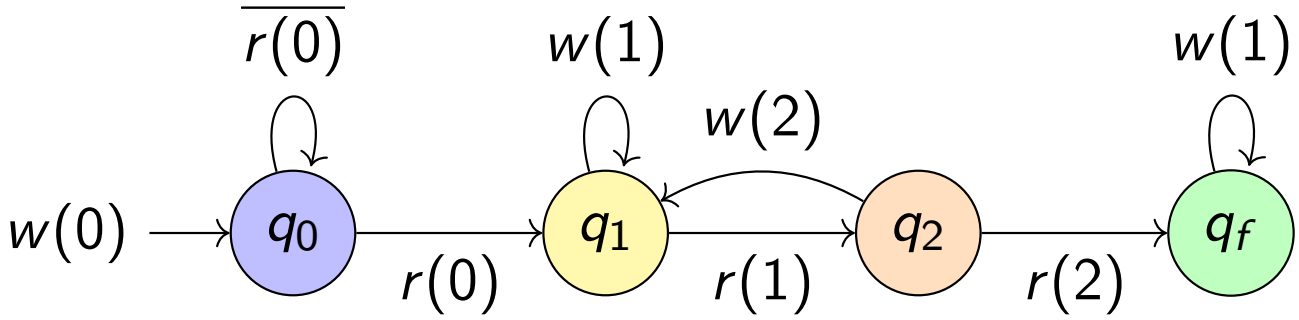
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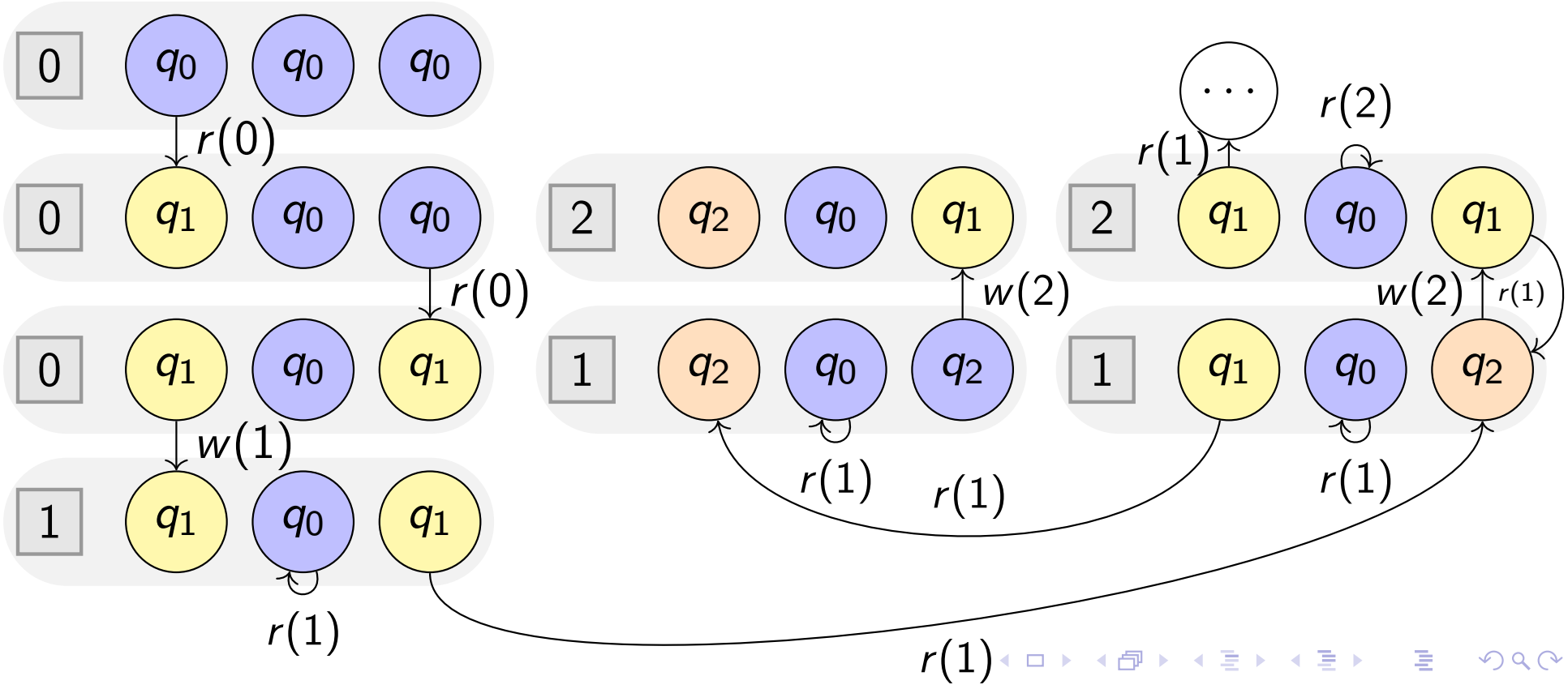
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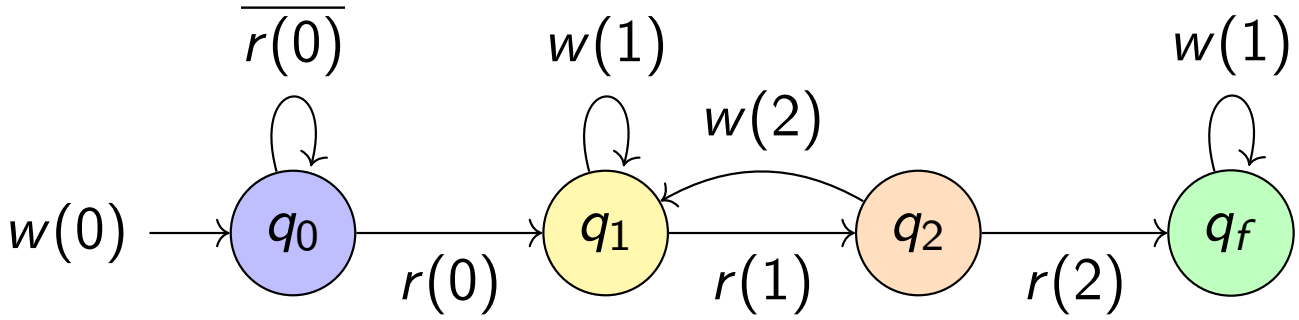
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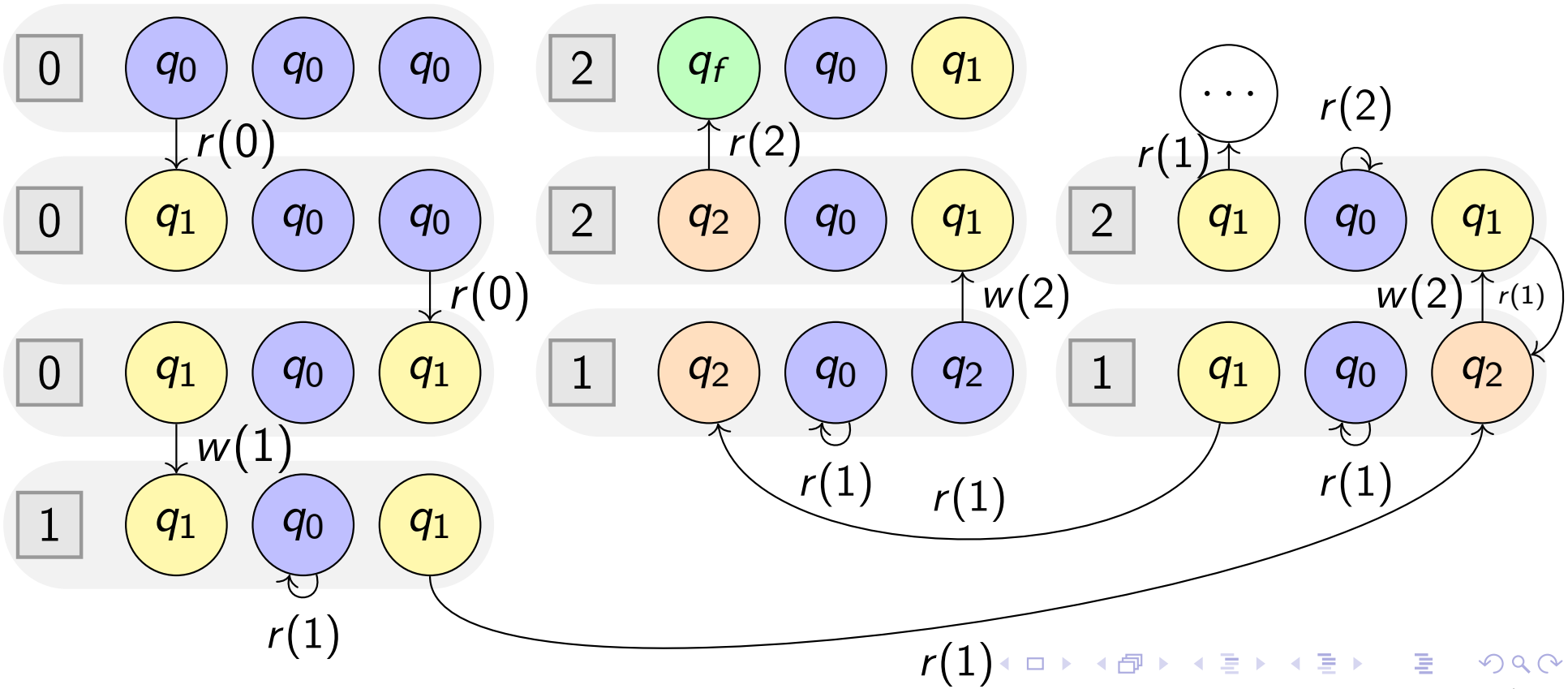
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1 The model

2 (Non-)Deterministic Model Checking
- Symbolic graph
- Reconstructing the symbols
- Conclusion

3 Probabilistic case

Definition (Reachability problem)

Let $(q_0, d_0) \in Q \times D$ and some target $q_f \in Q$. Does there exist $\gamma \in \Gamma$ with $\gamma(q_f) > 0$ reachable from $(q_0^{|\gamma|}, d_0)$?

Definition (Reachability problem with leader)

Let $(q_0, d_0) \in Q \times D$, $q_l \in Q$ and some target $q_f \in Q$. Does there exist $\gamma \in \Gamma$ with $\gamma(q_f) > 0$ reachable from $(q_l + q_0^{|\gamma|}, d_0)$?

- Once γ is fixed, the number of processes in the run is fixed.
- Monotonicity : if q_f is reachable with n processes, still reachable with a bigger number of processes.
- Bound of the maximal parameter value to consider ?

Symbolic graph

In the following, we consider the leader-less case.

Definition (Symbolic graph)

$G_{\text{symb}} = (S, E)$ with

- $S = 2^Q \times D$
- E is defined by $(X_1, d_1) \to (X_2, d_2)$ if there exists $x_1 \in X_1$, $x_2 \in X_2$ such that

$$X_1 \backslash \{x_1, x_2\} = X_2 \backslash \{x_1, x_2\} \tag{1}$$

$$x_1 \xrightarrow{w(d_2)} x_2 \vee \left(x_1 \xrightarrow{r(d_2)} x_2 \wedge d_1 = d_2 \right) \tag{2}$$

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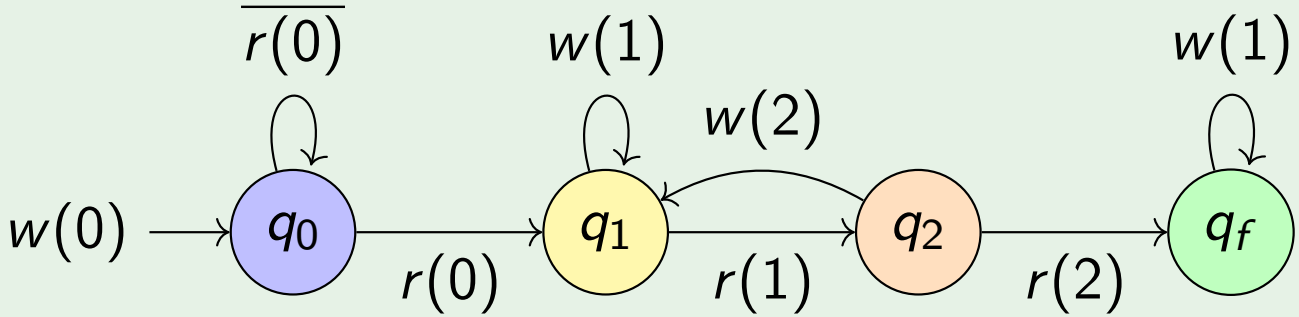
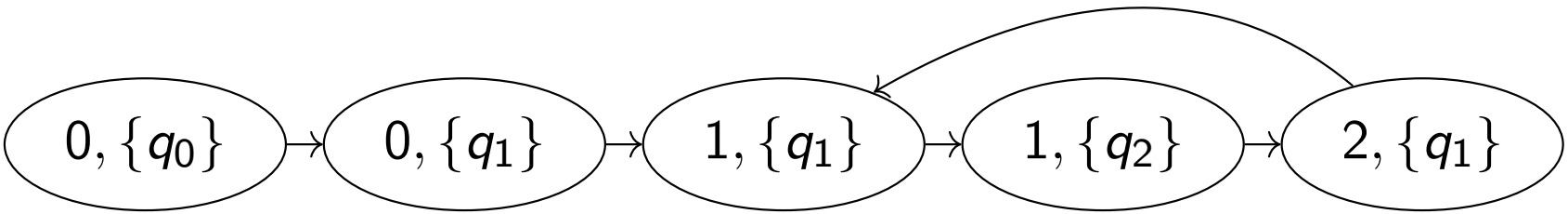
Lemma

Every "concrete" run of $\mathcal{P}$ corresponds to a symbolic run. $\overline{S}(\cdot)$ can be seen as an abstraction.

Example





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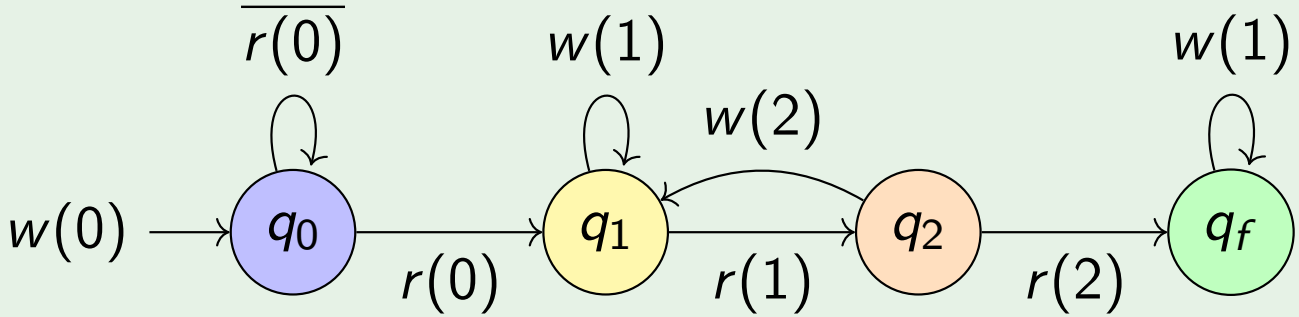
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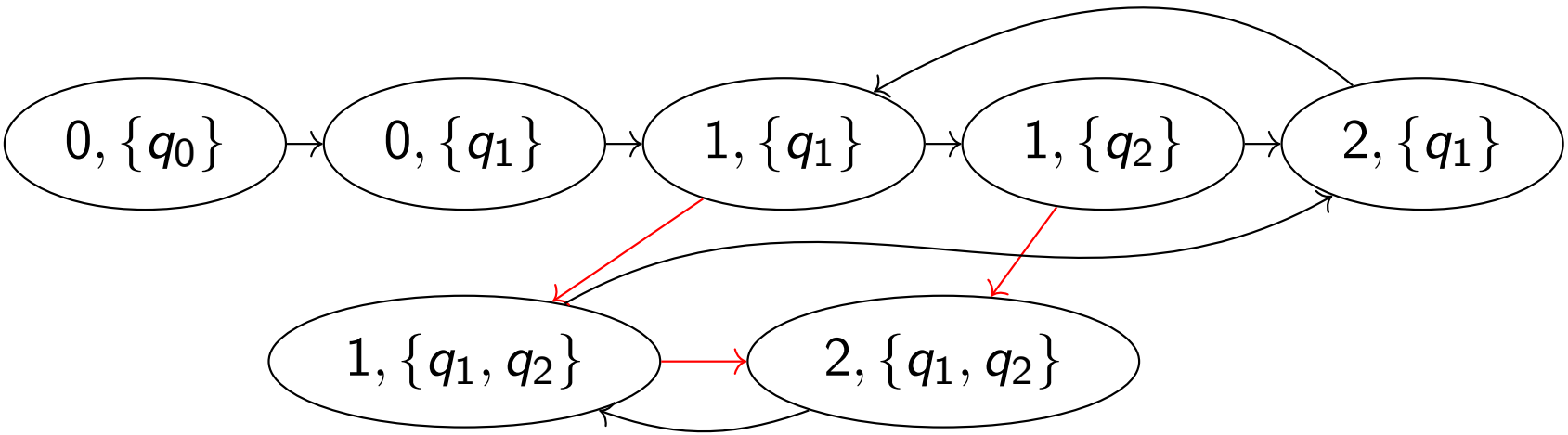
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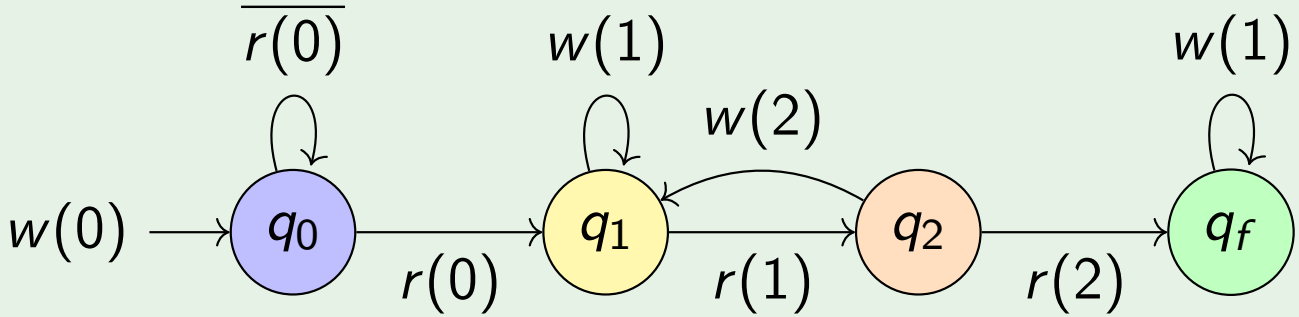
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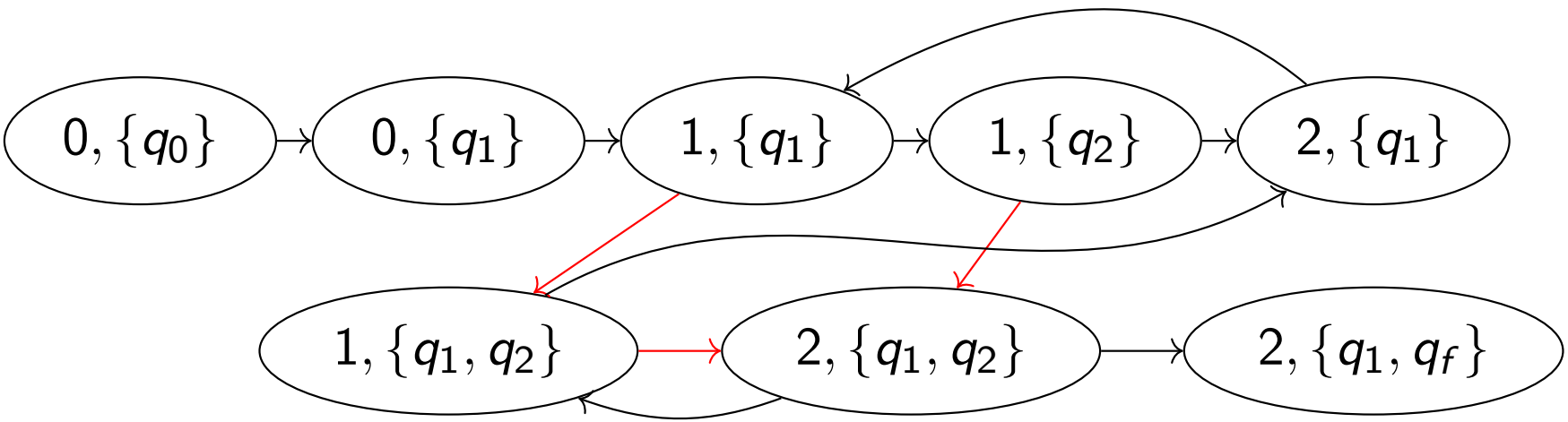
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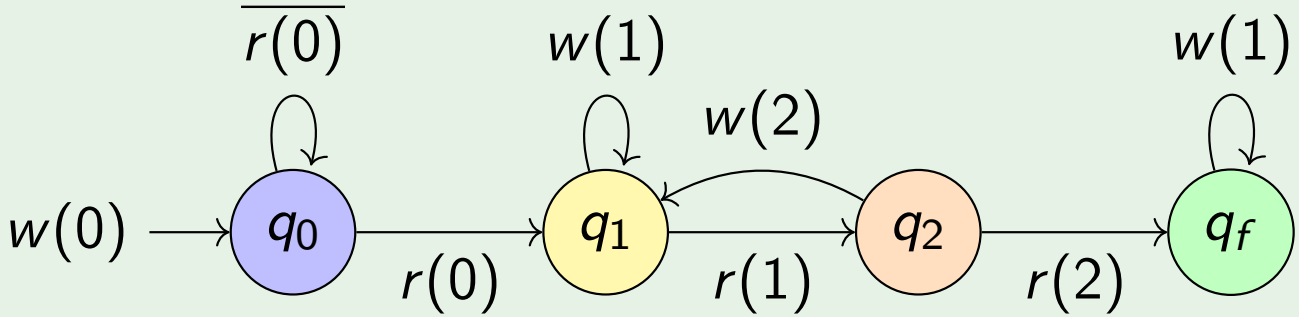
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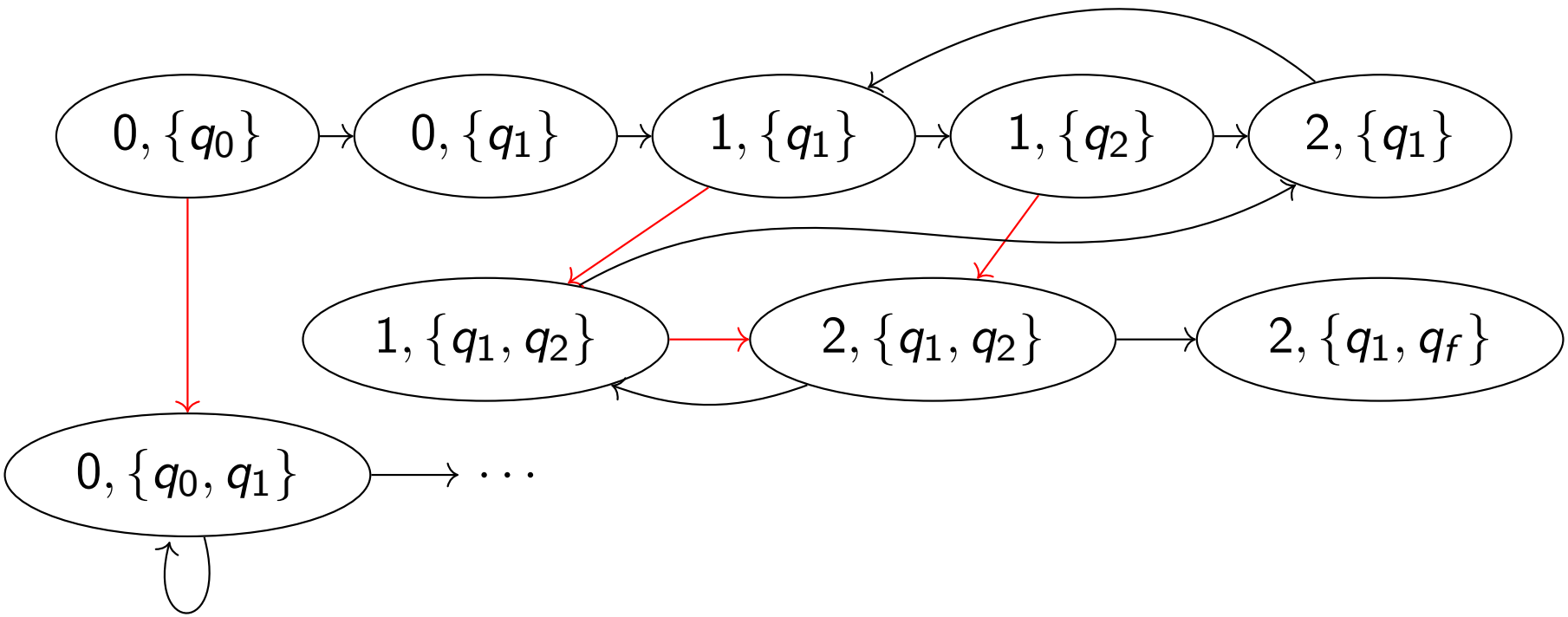
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$1, \{q_2\}$
$2, \{q_1\}$
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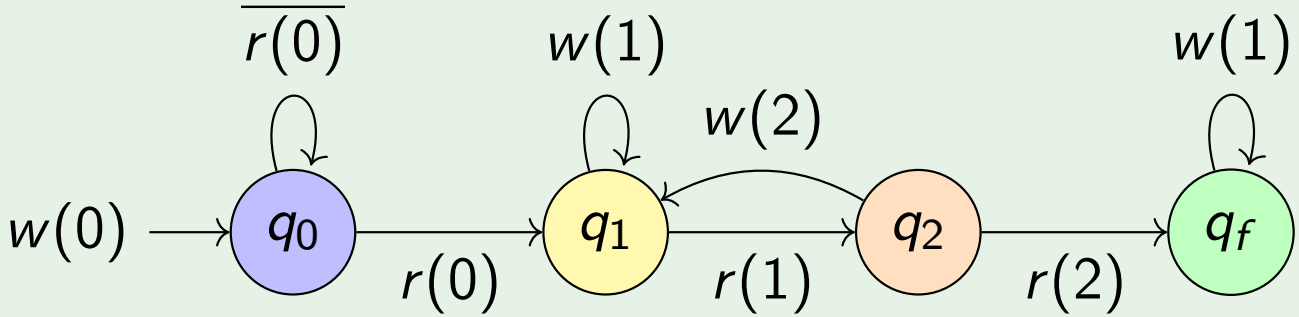
$\cdots$

Example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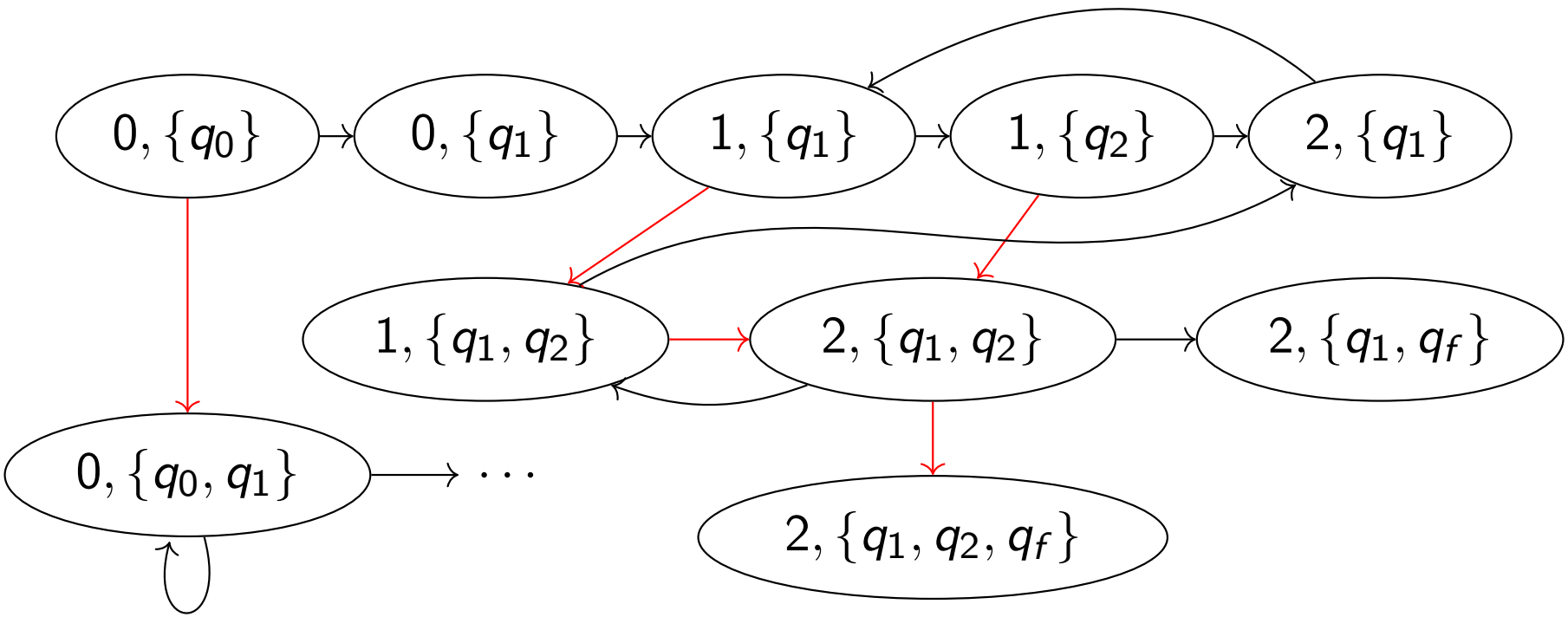
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Reconstructing the run

Lemma

If $(X, d) \rightarrow^{L} (Y, e)$ *in* G_{symb}*, then there exists a concrete run* $\gamma \rightarrow^{*} \gamma'$ *with*

- $\overline{S}(\gamma) = X$
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Sketch.

Every red transition in the symbolic graph implies a copy of the current involved state. □

Reducing a symbolic path

$$X_1, d_1 \longrightarrow X_k, d_k$$

Reducing a symbolic path

Reducing a symbolic path

Reducing a symbolic path

Reducing a symbolic path

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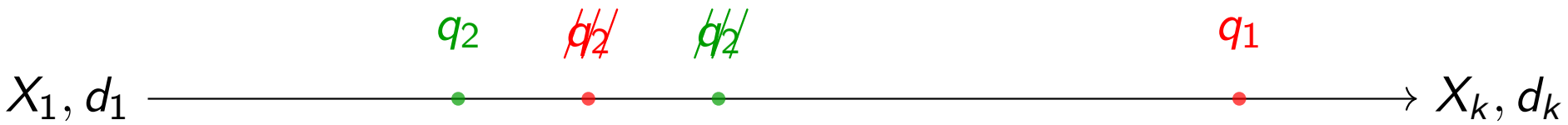
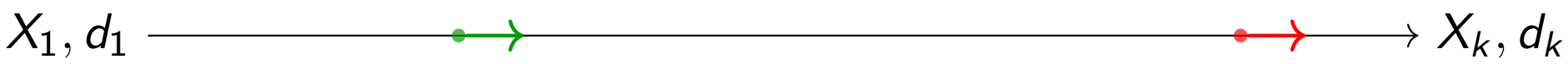


Reducing a symbolic path

Without loss of generality

For each $q \in Q$, there is at most two transitions making q appearing or disappearing (one each).

Fixed support behaviour



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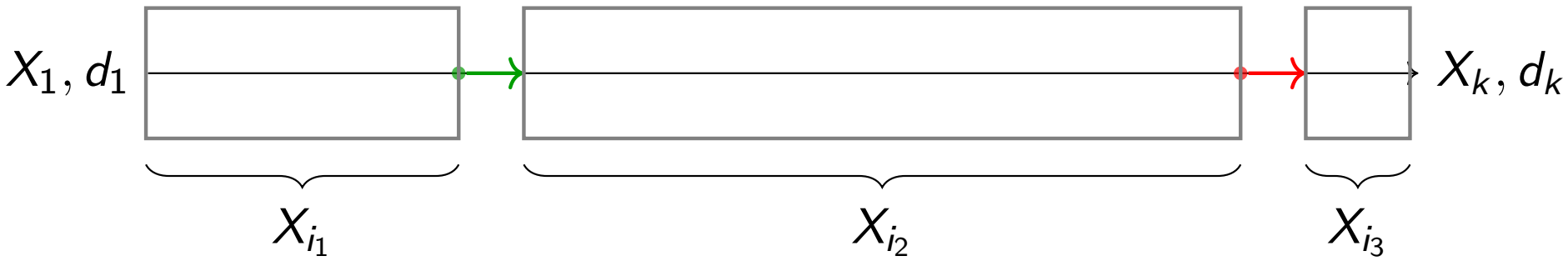

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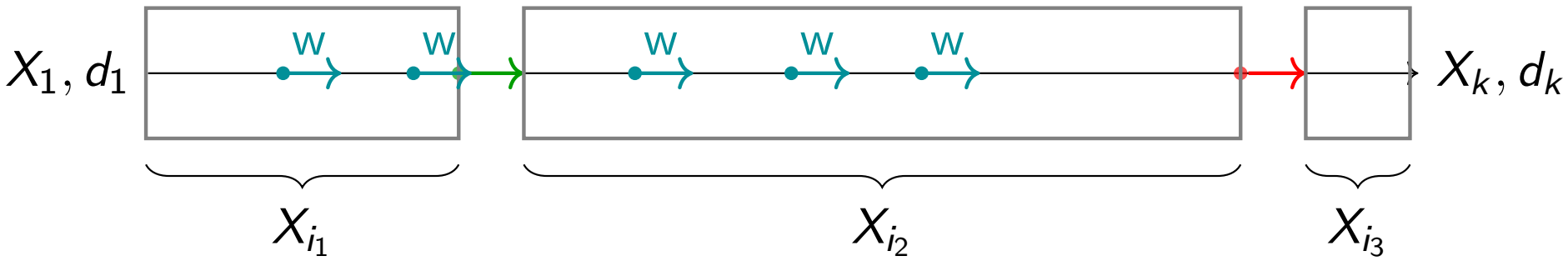
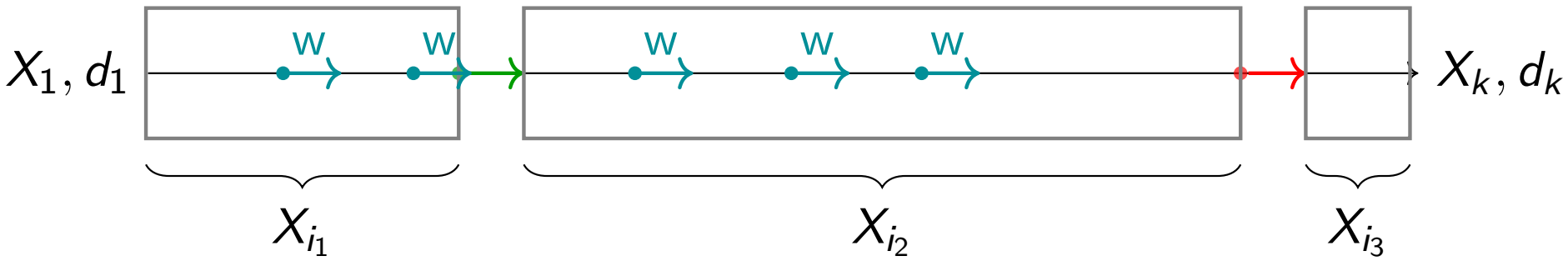


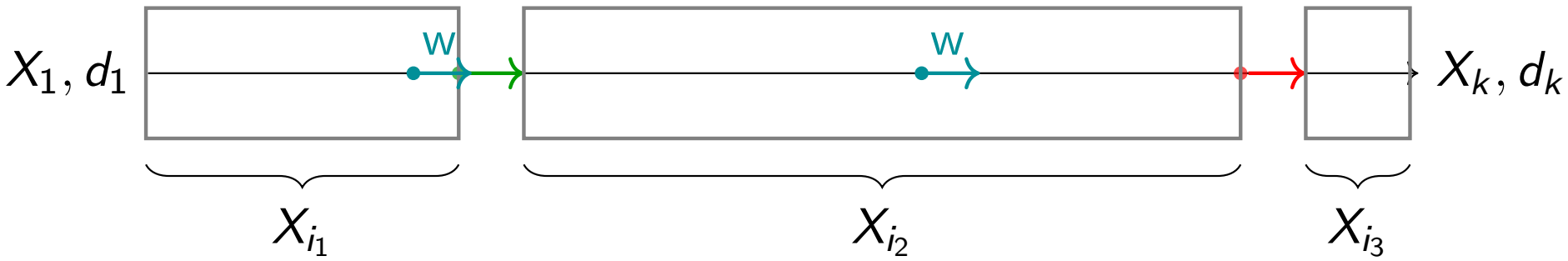
Fixed support behaviour



Key Idea

Only the last write transition is required.

Fixed support behaviour



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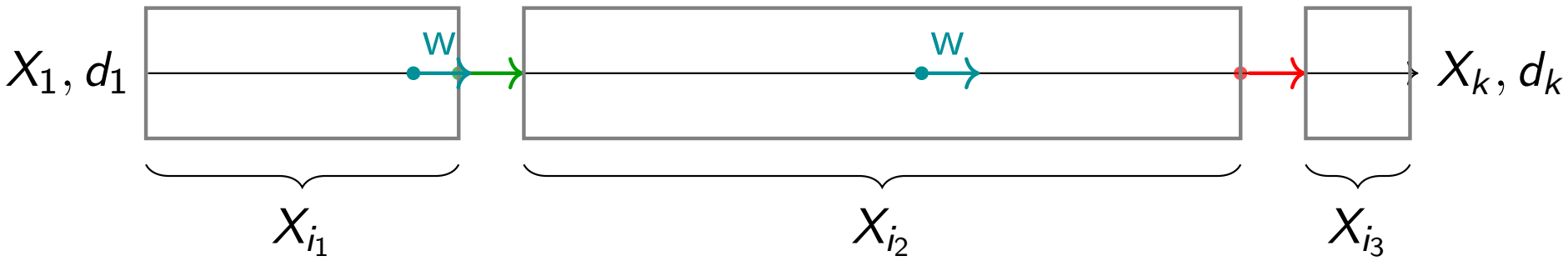

Key Idea

Only the last write transition is required.

Theorem

Every path in G_{symb} *has less than* $4|Q| + 1$ *transitions.*

What the symbolic graph taught us

Theorem

If $\gamma \rightarrow^* \gamma'$ there exists $\eta \rightarrow^* \eta'$ with same set of states/register values such that $|\eta| \leq 4|Q| + 1$.

Leader case

In case of a leader, we have to keep track of the exact control state of the leader process.

- Symbolic states : (X, q'_l, d)
- Still $2|Q| + 1$ fixed support blocks
- Each block can be reduced to last write and at most $|Q|$ transitions of the leader
- Final bound of the form $2|Q| + (2|Q| + 1)|Q|$.

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Theorem

The reachability problem is NP-complete.

Sketch.

- Guess the polynomial path.
- Reduction from 3−SAT.

□

Language formalism

Definition

$$L(\mathcal{P}) = L(\mathcal{A}_D) \cap \left(L(\mathcal{A}_I) \,\text{⧔}\, \bigcup_{k \geq 1} \,\text{⧔}_{i=1}^{k}\, L(\mathcal{A}) \right)$$

Where

- $L(\mathcal{P})$ is the language of all possible infinite runs.

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- $L(\mathcal{P})$ is the language of all possible infinite runs.
- The automata recognize languages over $\Sigma_D = \{w(d), r(d) \mid d \in D\}$.
- $\between$ is the shuffle operations over languages.
- $\mathcal{A}_D$ is an automaton recognizing correct sequences of read and write operations.
- $\mathcal{A}_l$ is the automaton of the leader.
- $\mathcal{A}$ is the automaton of the protocol.

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Given a Bchi automaton $\mathcal{A}_\varphi$, decide whether $L(\mathcal{P}) \cap L(\mathcal{A}_\varphi) = \emptyset$.

1 The model

2 (Non-)Deterministic Model Checking

3 Probabilistic case
- Probabilistic semantics
- Cut-off property
- Existence of a cut-off
- Complexity aspects
- A linear example
- PSPACE hardness
- Upper Bound

Markov Chain

Definition (Law of motion)

We consider $(\Gamma, \rightarrow)$ as a Markov Chain.

$$\Pr(\gamma \rightarrow \gamma') = \frac{1}{|\mathrm{Post}(\gamma)|}$$

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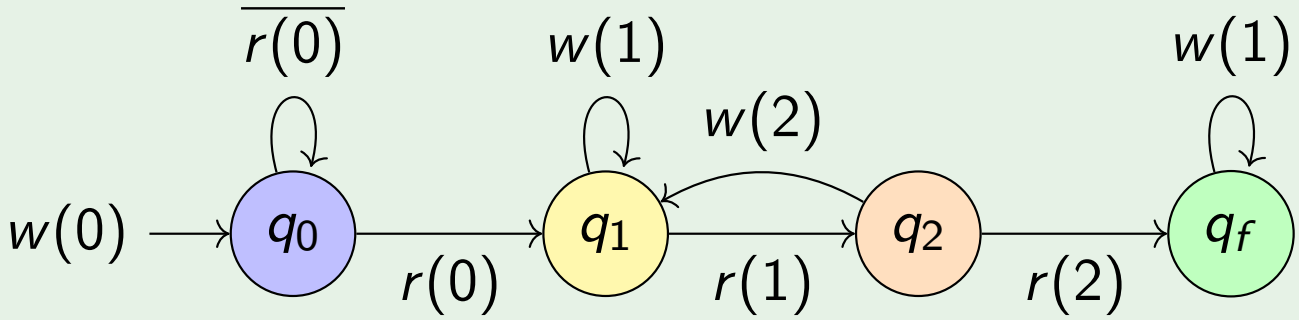
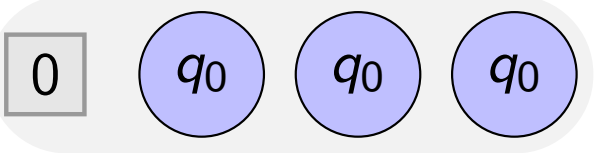
Qualitative goal

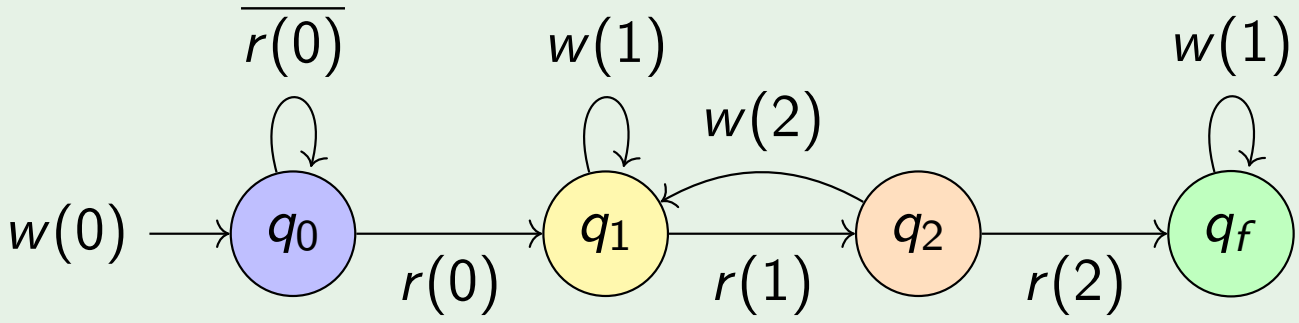
Let $q_f \in Q$.
Estimate $\mathbb{P}^n(\uparrow q_f)$

Example





Example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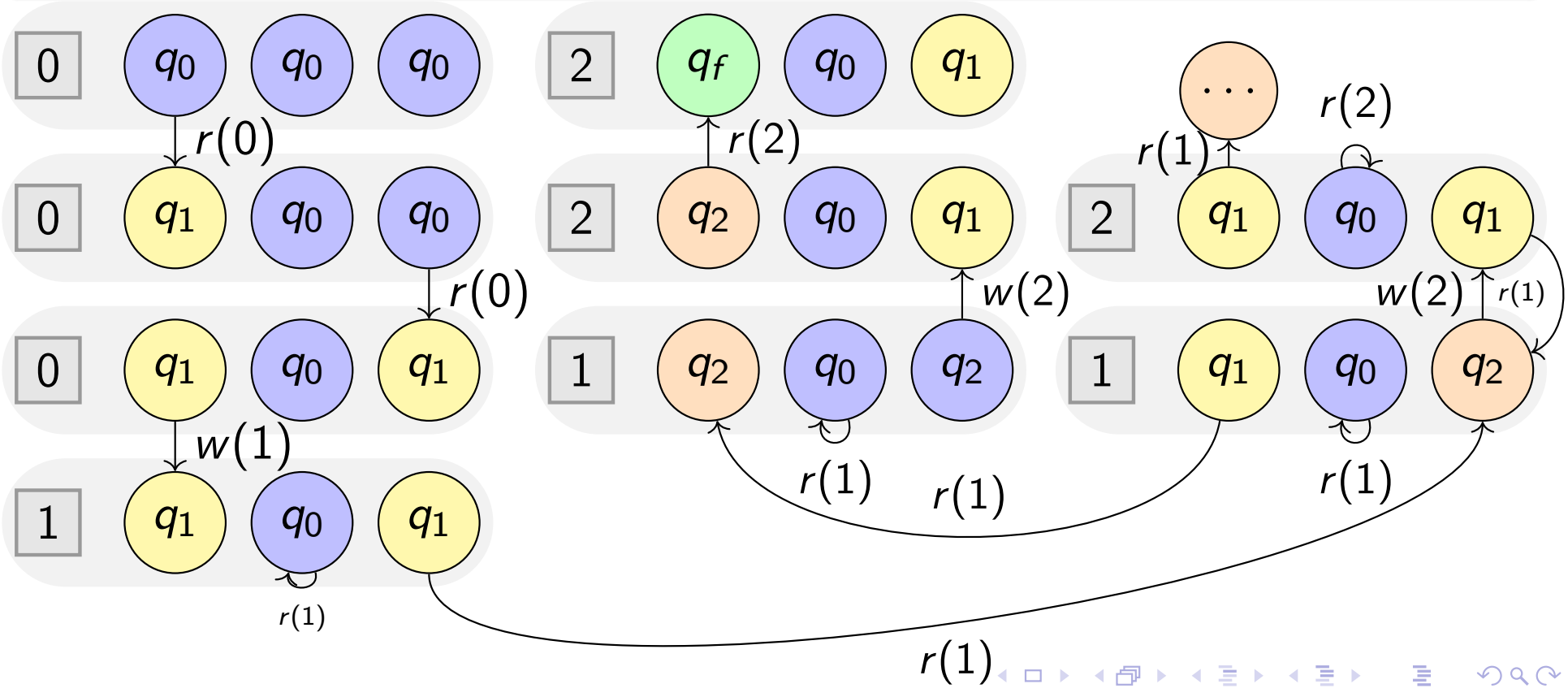
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$r(1)$
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q_f q_0 q_1
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q_2 q_0 q_1
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1
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$\cdots$
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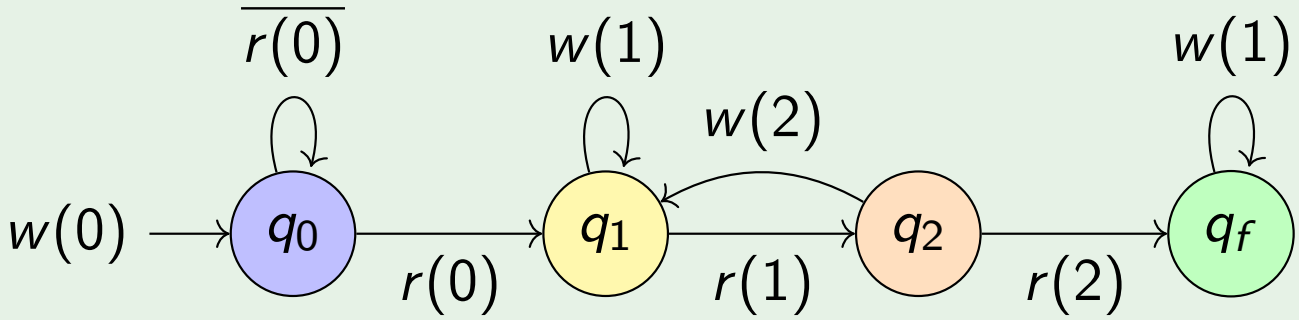
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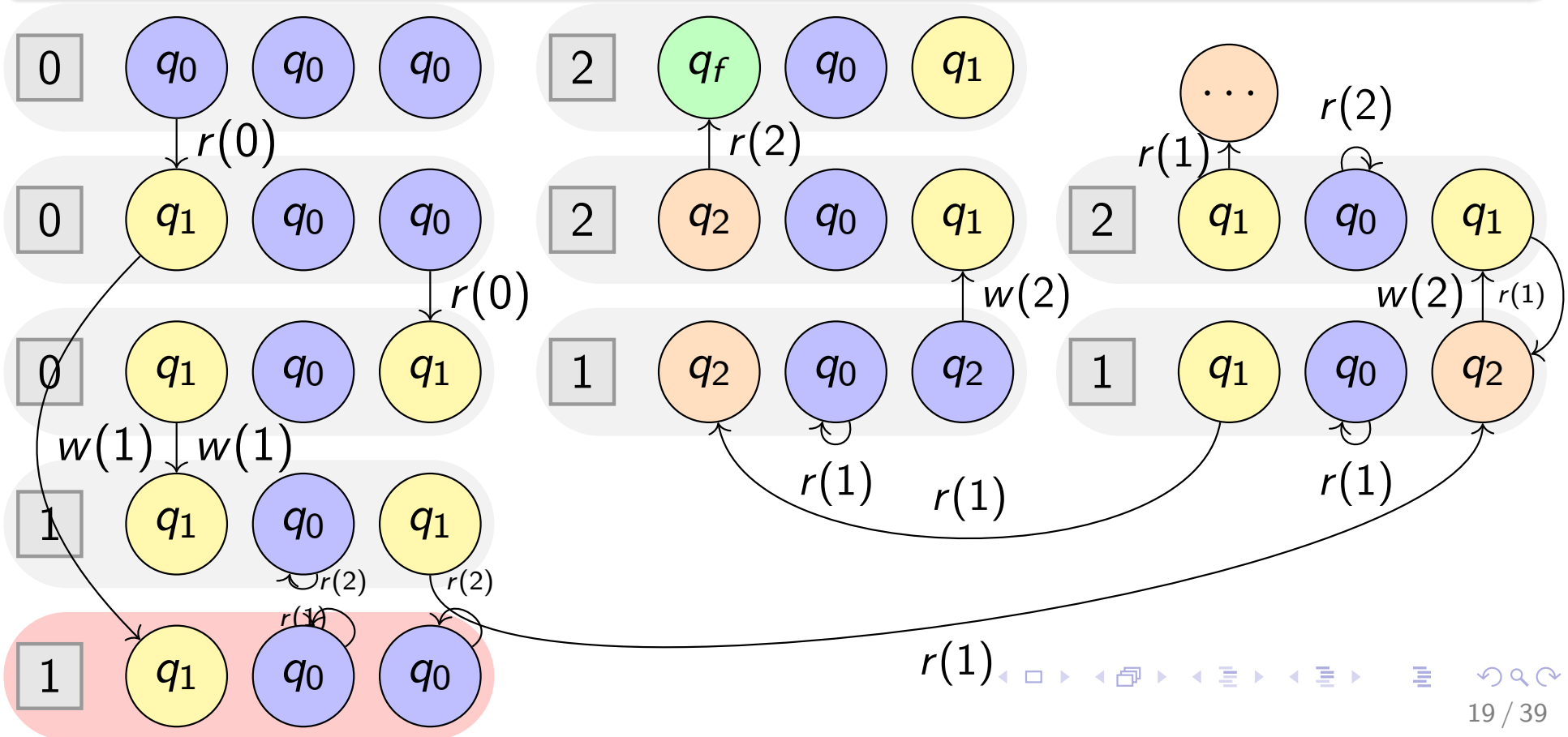
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Remarks

Lemma (Qualitative assumption)

The properties $\mathbb{P}^n(\uparrow q_f) > 0$ *and* $\mathbb{P}^n(\uparrow q_f) = 1$ *does* <u>not</u> *depend on the actual probabilities.*

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- We focus now on the almost-sure ($\mathrm{Pr}^n(\uparrow q_f) = 1$ problem).
- Both the scheduler and processes are ~~non-deterministic~~ stochastic
- No atomicity
- No monotonicity a priori
- For fixed parameter n, can be encoded as a Petri Net

What we are looking for

Some limit behaviour, if possible

Definition (Cut-off)

Le N a parameter. If $\forall n \geq N \ \mathbb{P}^n(\uparrow q_f) = 1$ or $\forall n \geq N \ \mathbb{P}^n(\uparrow q_f) < 1$, then N is a cut-off.

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- First case is said positive, second case negative
- Non-atomicity is crucial

Upward closed sets and well-quasi-orders

Definition (Well-quasi-ordering (wqo))

A quasi-ordering $\preceq$ on a set X such that any infinite sequence of elements $x_0, x_1 \ldots$ from X contains an increasing pair $x_i \preceq x_j$ with $i < j$.

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Let $Y \subseteq X$, such that $\forall \gamma \preceq \gamma' \ \gamma \in Y \Rightarrow \gamma' \in Y$, then Y is generated by $\min(Y)$*:*

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Main idea: express $\mathrm{Pre}^*(\uparrow q_f)$ and $\mathrm{Post}^*((q_0^k, d_0))$ as upward-sets to "discretize" the problem.

The right partial order

Definition

$$\gamma \preceq \gamma' \Leftrightarrow v(\gamma) = v(\gamma') \wedge \forall q \; \gamma(q) \leq \gamma'(q)$$

$\preceq$ is a well quasi-order.

- $\mathrm{Pre}^*(\uparrow q_f)$ is upward-closed
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Lemma (Mimicking process)

Let $(q_0^k, d_0) \to^* \gamma$ *and* q *such that* $\gamma(q) > 0$. *Then,* $(q_0^{k+1}, d_0) \to^* \gamma + q$.

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Cut-off

We write

$$\mathrm{Post}^*(\uparrow (q_0, d_0)) = \cup_{i=1}^{k} \uparrow \gamma_i \quad \mathrm{Pre}^*(\uparrow q_f) = \cup_{j=1}^{k} \uparrow \eta_j$$

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What about the negative cut-off case ?

Negative case

$$\mathrm{Post}^*(\uparrow (q_0, d_0)) = \cup_{i=1}^{k} \uparrow \gamma_i$$

- If there exists γ_i such that $\forall j\ \overline{S}(\gamma_i) \neq \overline{S}(\eta_j)$, then ...

Negative case

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How does

$$\min \mathrm{Post}^*(\uparrow (q_0, d_0)) = \{\gamma_{1,n} \dots \gamma_{k_n,n}\}$$

varies w.r.t n ?

$\mathrm{Post}^*(\uparrow(q_0, d_0))$ γ_1 γ_2 γ_3 $\cdots$ γ_k

$\mathrm{Post}^*(\uparrow(q_0, d_0))$ γ_1 γ_2 γ_3 $\cdots$ γ_k

$n+1$


$\mathrm{Post}^*(\uparrow(q_0, d_0))$
γ_1
γ_2
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$\cdots$
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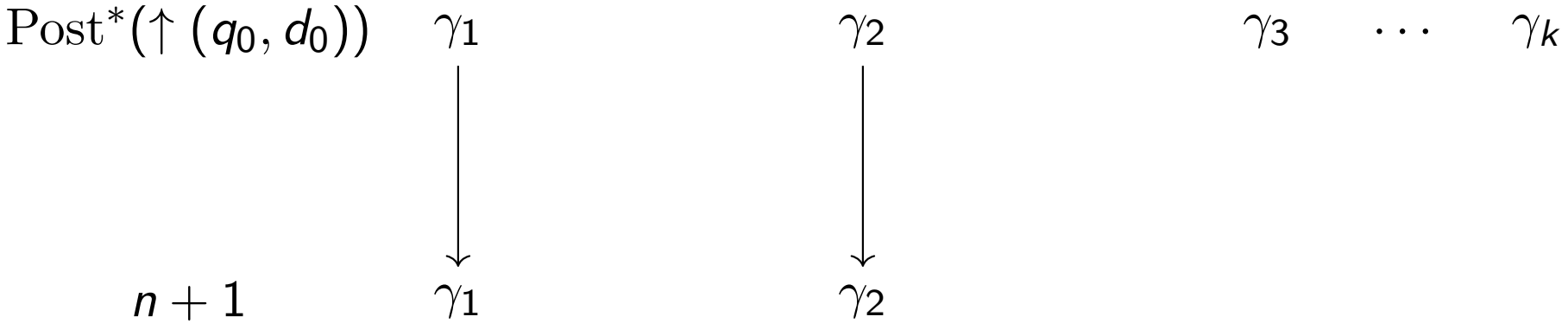
$n+1$
γ_1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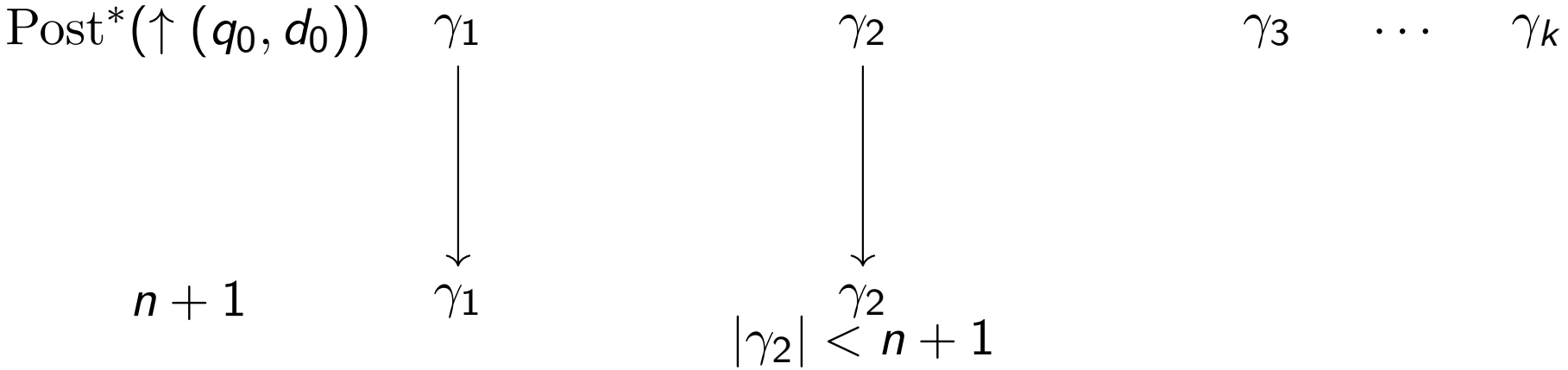
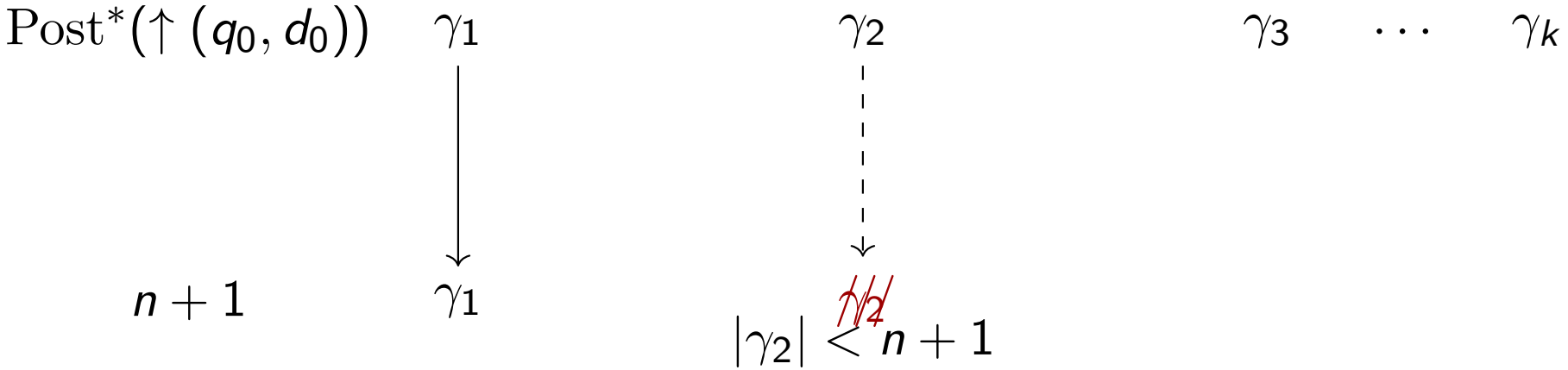
γ_1
γ_2


$\mathrm{Post}^*(\uparrow(q_0, d_0))$
γ_1
γ_2
γ_3
$\cdots$
γ_k
$n+1$
γ_1
γ_2
$|\gamma_2| < n+1$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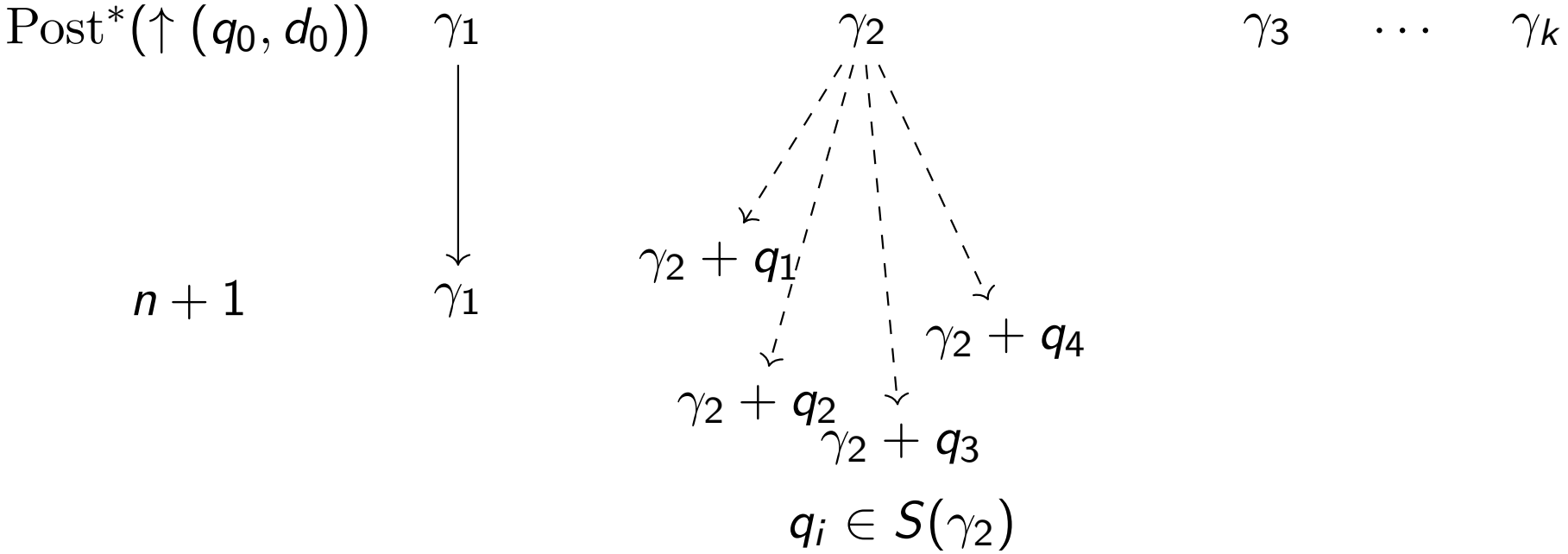
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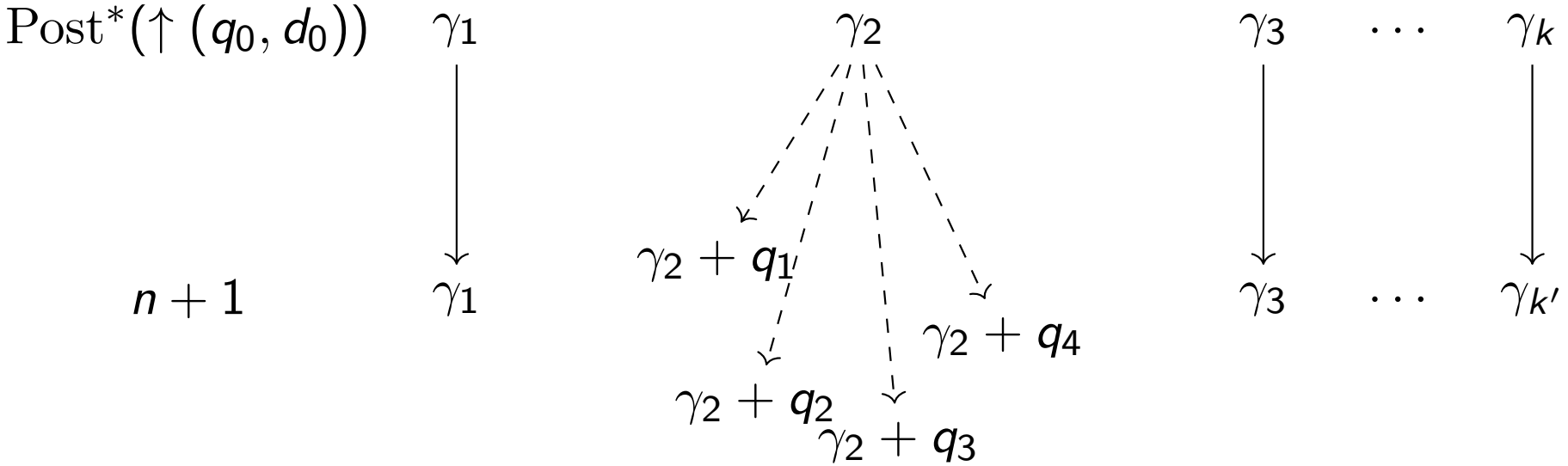
γ_1
$\gamma_2 + q_1$
$\gamma_2 + q_4$
$\gamma_2 + q_2$
$\gamma_2 + q_3$
$q_i \in S(\gamma_2)$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γ_1
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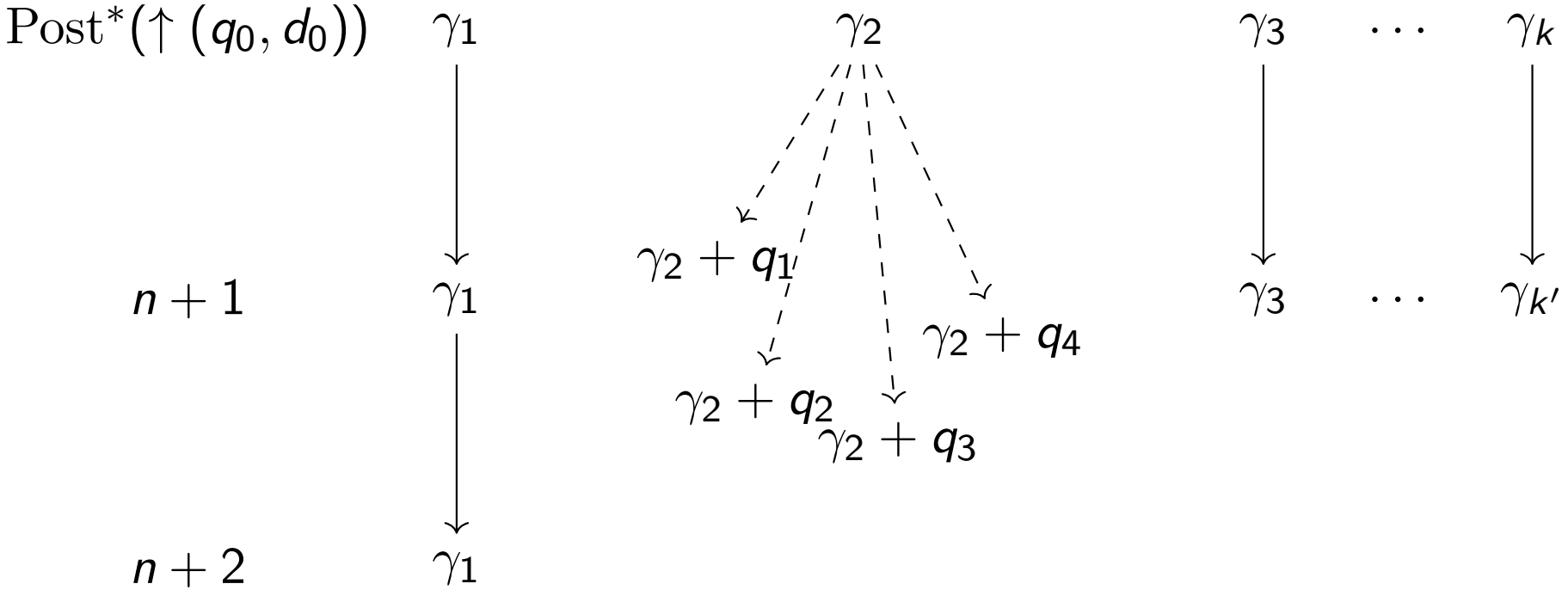
$\cdots$
$\gamma_{k'}$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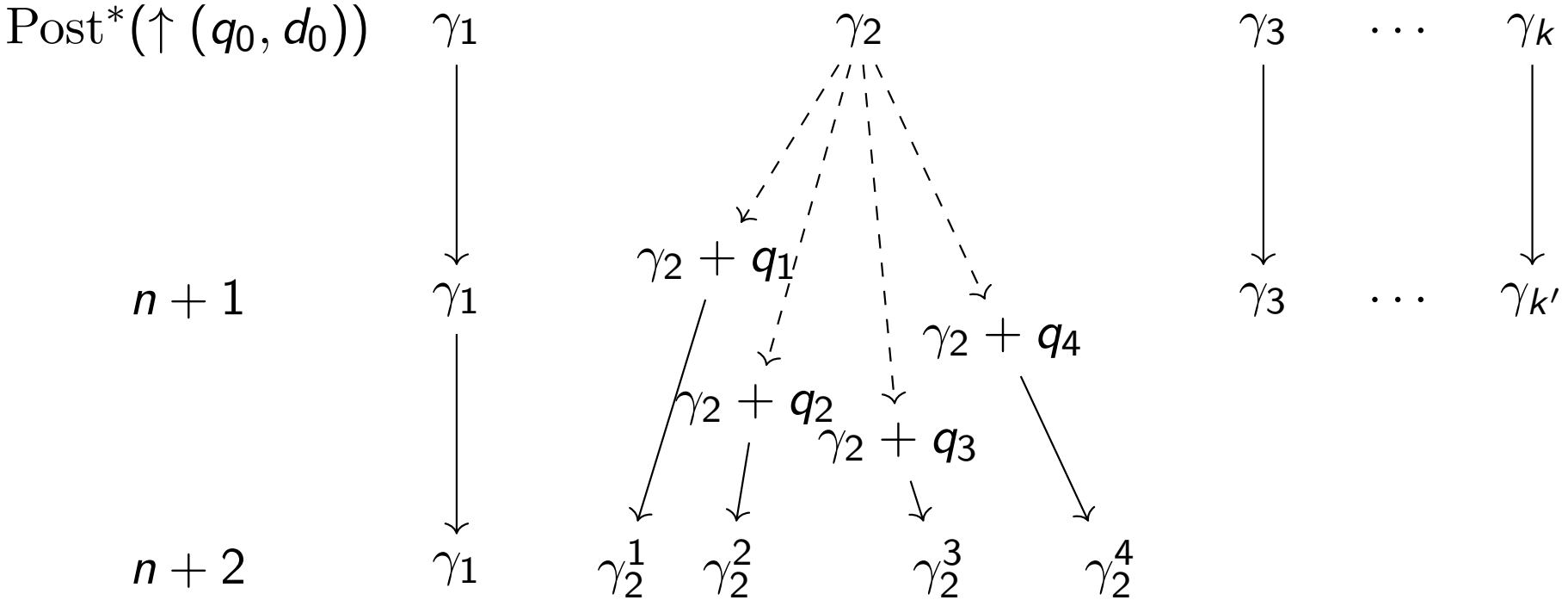
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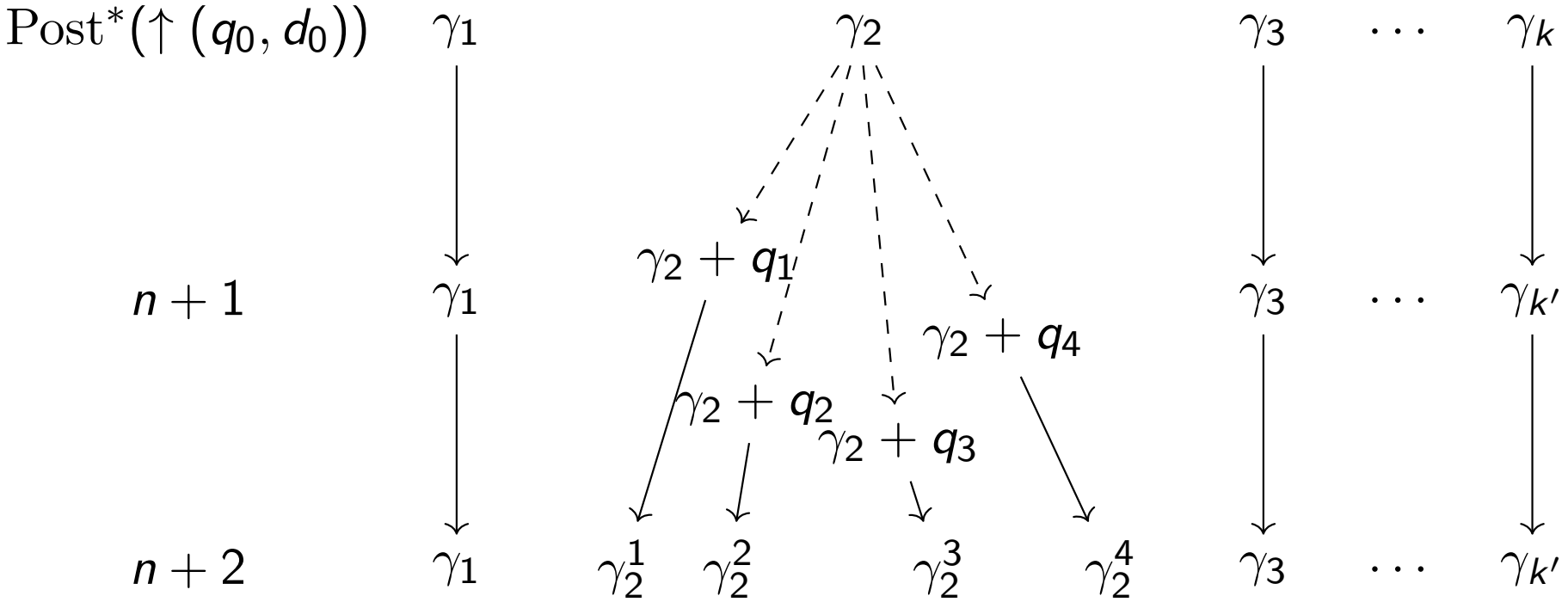
$\gamma_{k'}$
$n+2$
γ_1
γ_2^1
γ_2^2
γ_2^3
γ_2^4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1. Start with $X_1 = \{\gamma_1 \dots \gamma_k\} = \min \mathrm{Post}^*(\uparrow (q_0, d_0))$

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Goal

Keep track of a "problematic" generator $\gamma \in X_k$ as $k \to \infty$.

Definition

Assume

$$\mathrm{Pre}^*(\uparrow q_f) = \cup_{j=1}^{k} \uparrow \eta_j$$

and let $\gamma \in \mathrm{Post}^*(\uparrow (q_0, d_0))$. *We define:*

$$A(\gamma) = \bigcap_{j=1}^{m} \left\{ q \in S(\gamma) \;\middle|\; \forall k \geq 0 \; \eta_j \not\preceq \gamma + q^k \right\}$$

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Theorem (Negative cut-off)

If $A(\gamma) \neq \emptyset$. *Then* $N = |\gamma|$ *is a negative cut-off.*

Positive cut-off

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Assume $A(\gamma) = \emptyset$. *Then there exists* $N(\gamma) \geq |\gamma|$ *such that*

$$(\uparrow \gamma) \cap \mathrm{Post}^* \left(q_0^{\geq N(\gamma)}, d_0 \right) \subseteq \mathrm{Pre}^*(\uparrow q_f)$$

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Proof.

For any $q \in S(\gamma)$, there exists j_q such that $q \notin \left\{ q \;\middle|\; \forall k \geq 0 \; \eta_{j_q} \not\preceq \gamma + q^k \right\}$ so there exists k_q such that $\eta_{j_q} \preceq \gamma + q^{k_q}$.
Define $N(\gamma) = |\gamma| + \sum_{q \in S(\gamma)} k_q$. □

Existential solution

Theorem

Given a protocol $\mathcal{P}$ there always exists either a positive cut-off either a negative cut-off N.
The probability to reach $\uparrow q_f$ is eventually 1 *or eventually strictly less than* 1.

- Non-constructive proof
- We never computed the γ_i and η_j
- If computed, we can give a polynomial bound of N in their size
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- Simulate the Markov chain with N initial processes ?

Negative cut-off: the easy case

Remark

If $\mathrm{Post}^*((\{q_0\}, d_0)) \not\subseteq \mathrm{Pre}^*(\uparrow q_f)$ in G_{symb}, then $\mathbb{P}^n(\uparrow q_f) < 1$ for n large enough (negative cut-off).

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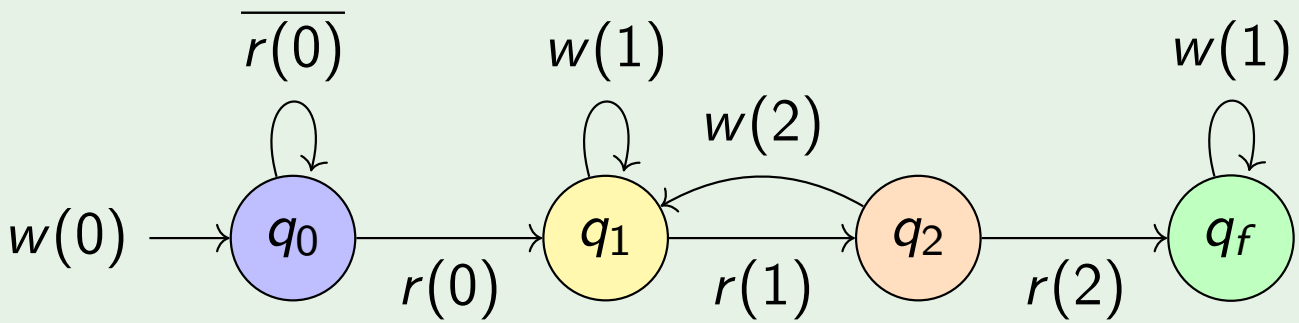

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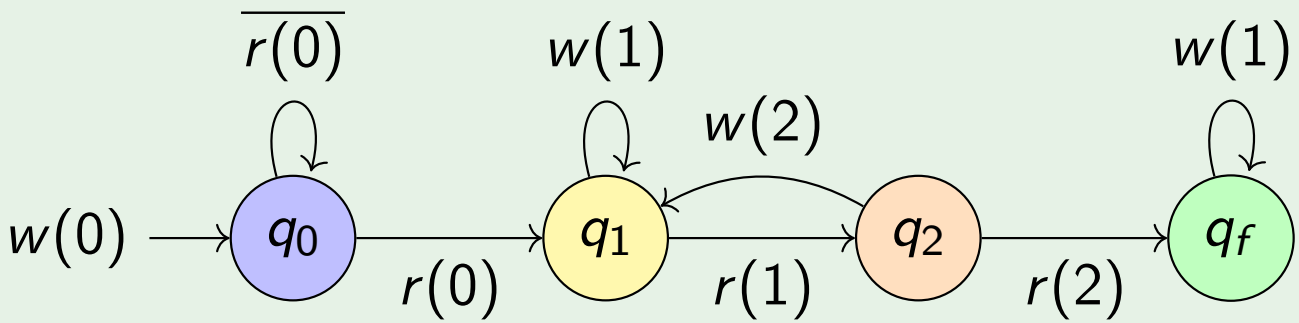
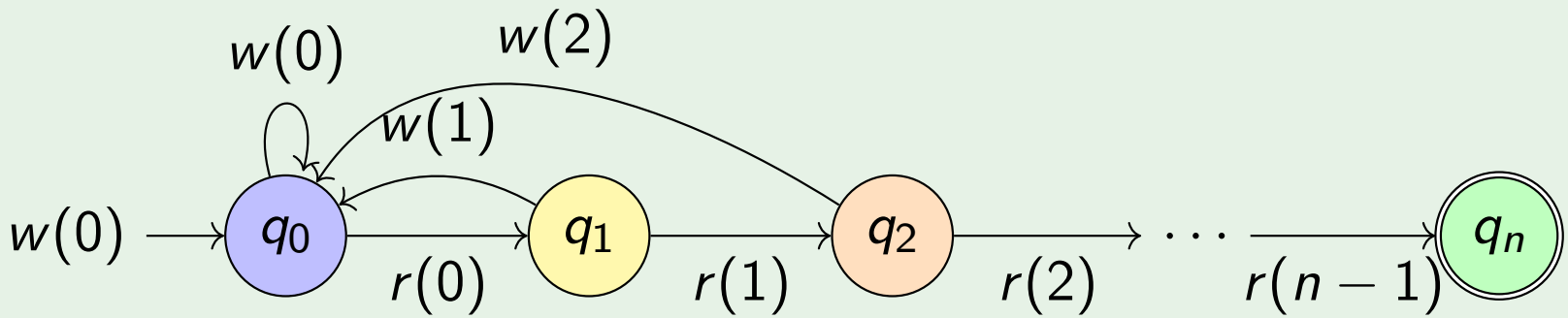


$$(q_0^n, 0) \xrightarrow{r(0)} (q_0^{n-1} q_1, 0) \xrightarrow{w(0)} (q_0^{n-1} q_1, 1) \not\to^* \uparrow q_f$$

Linear example

Example



Cut-off value ?

Linear example

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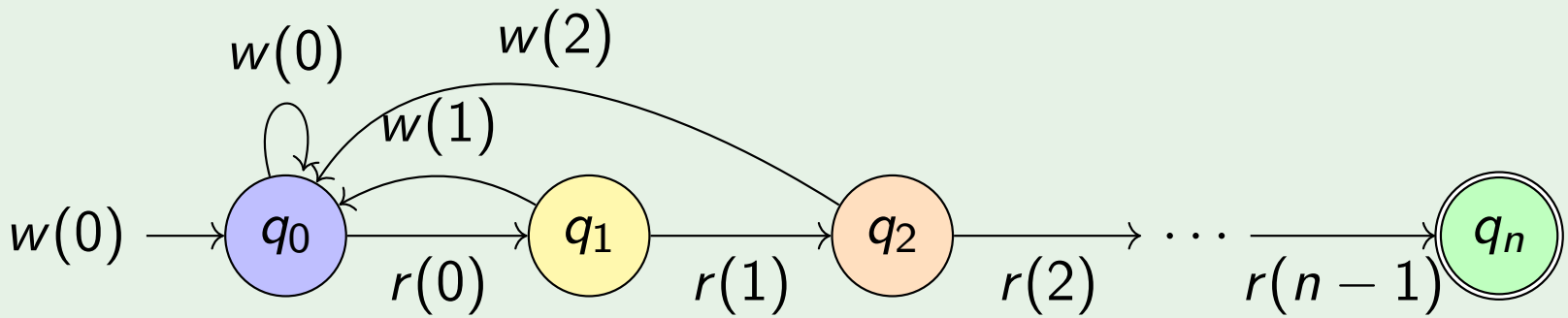

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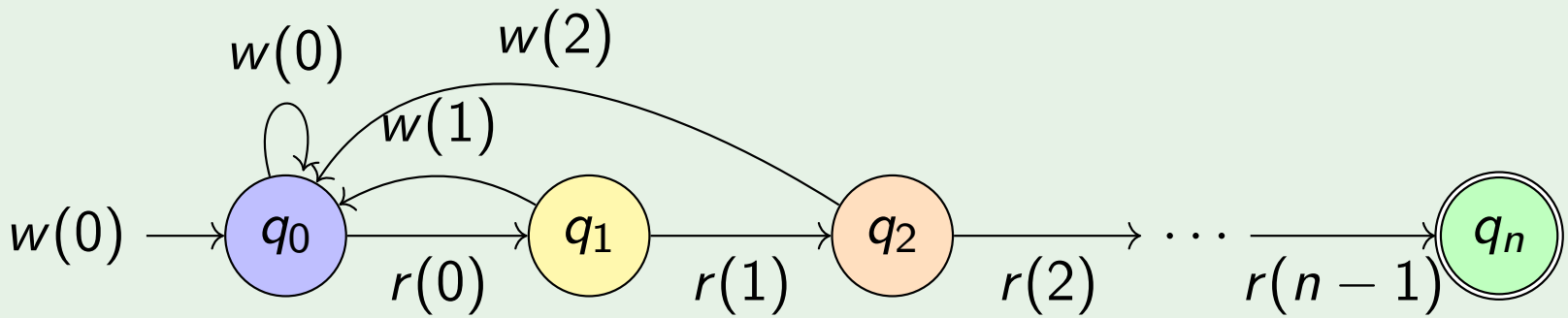

Cut-off value ? The cut-off is positive and equals n.

Invariant:

$$\forall j \leq m \ \sum_{k=0}^{j} \gamma(q_k) \geq j + \mathbb{1}_{v(\gamma)=j+1}$$

PSPACE-hardness

Decision Problem

- INPUT: a protocol $\mathcal{P}$
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Theorem

The cut-off decision problem is PSPACE-hard.

Sketch.

We reduce the halting of a linear bounded Turing machine $\mathcal{M}$.

- A given tape position i containing letter x is coded by a fixed state
- The current head position is coded in the register

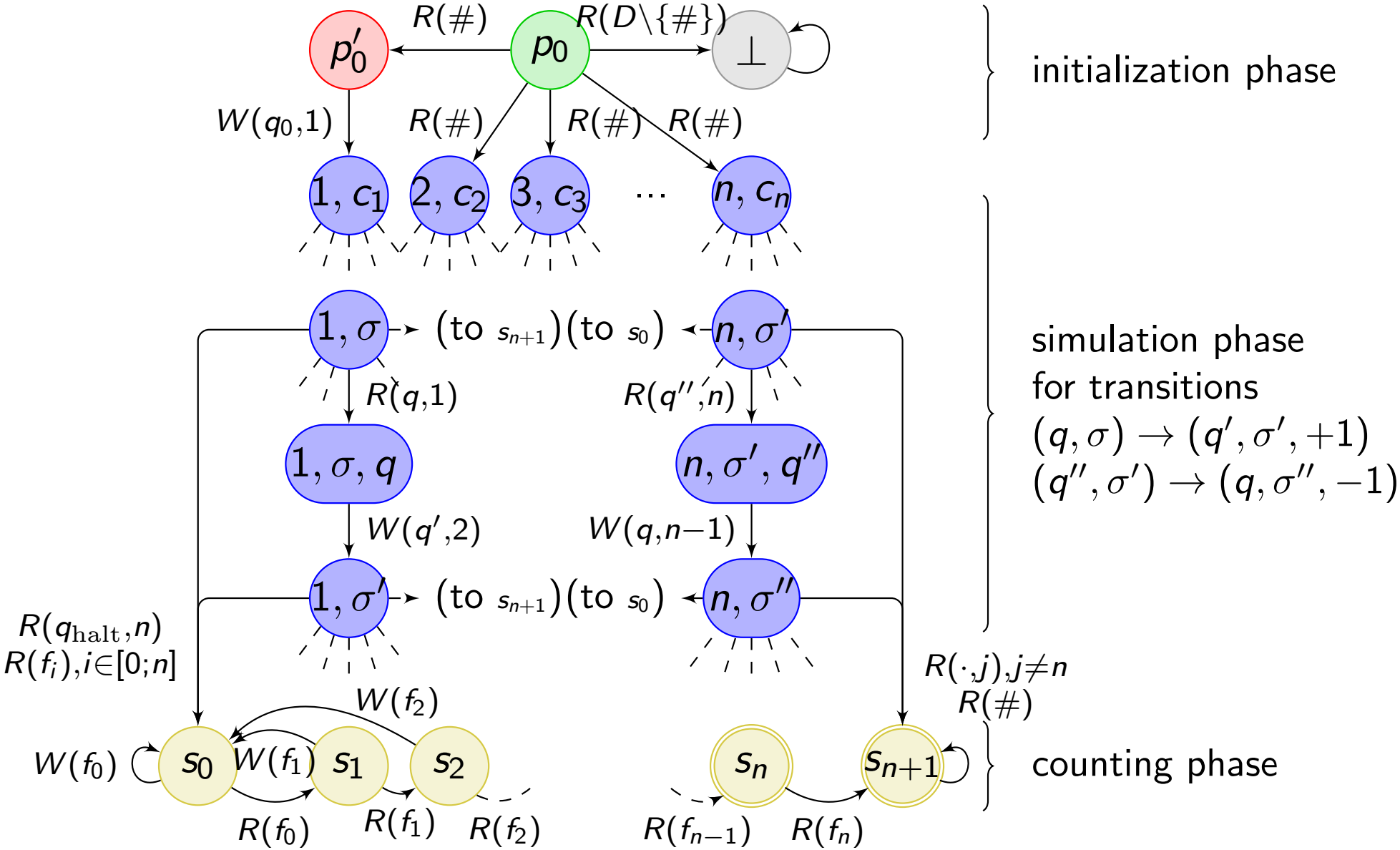
Sketch.

We reduce the halting of a linear bounded Turing machine $\mathcal{M}$.

- A given tape position i containing letter x is coded by a fixed state
- The current head position is coded in the register
- If the number of states is too big, we cannot ensure proper encoding
- Key idea: we code improper encoding/non-termination by $\mathbb{P}(\uparrow q_f) = 1$
- The previous module ensures $\uparrow q_f$ if too many processes encoded the machine
- We assume the machine starts from first tape position and accepts only on last tape position.

We build $\mathcal{P}$ protocol such that

$$\mathbb{P}^{\geq n}(\uparrow q_f) = 1 \iff \mathcal{M} \text{ does } \underline{\text{not}} \text{ terminate}$$


p'_0
$R(\#)$
p_0
$R(D\backslash\{\#\})$
$\bot$
initialization phase
$W(q_0,1)$
$R(\#)$
$R(\#)$
$R(\#)$
$1, c_1$
$2, c_2$
$3, c_3$
$\dots$
n, c_n
$1, \sigma$
(to s_{n+1})(to s_0)
n, σ'
simulation phase
for transitions
$(q, \sigma) \to (q', \sigma', +1)$
$(q'', \sigma') \to (q, \sigma'', -1)$
$R(q,1)$
$R(q'',n)$
$1, \sigma, q$
n, σ', q''
$W(q',2)$
$W(q,n-1)$
$1, \sigma'$
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n, σ''
$R(q_{\text{halt}},n)$
$R(f_i),i\in[0;n]$
$R(\cdot,j),j\neq n$
$R(\#)$
$W(f_2)$
$W(f_0)$
s_0
$W(f_1)$
s_1
s_2
s_n
s_{n+1}
counting phase
$R(f_0)$
$R(f_1)$
$R(f_2)$
$R(f_{n-1})$
$R(f_n)$

Consequences

- Negative cut-off can be exponential.
- No clue about positive cut-off.

Upper bound ?

- Existence theorem gives a cut-off polynomial in the size of the basis of $\mathrm{Post}^*(\uparrow (q_0, d_0))$ and $\mathrm{Pre}^*(\uparrow q_f)$.
- Rackoff's theorem: $\mathrm{Pre}^*(\uparrow q_f)$ can be bounded by M doubly-exponential in $|\mathcal{P}|$.
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- No bound on the Post^*.
- Idea: refine the symbolic graph to keep track of up to M processes in each state.
- Such graph is still doubly-exponential in $|\mathcal{P}|$
- Guessing a path can be done in EXPSPACE.

Summary and Perspectives

- Simple model but still non-trivial model
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- Simple model but still non-trivial model
- Non-atomicity ensures regularity hence decidability.

- Other properties (is $\uparrow q_f = \{(q_f^k, d) \mid k, d\}$ an harder property ?)
- More registers, leader process.
- Hardness result with atomic operations ?
- Strategies
- Reasonable Polynomial upper-bound ?

Thank you for your attention