



Mixed strategies in concurrent reachability games

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Barbizon

The deterministic game framework

Mixed strategies

Mixed Nash Equilibria

Concurrent non-zero sum games

- ▶ Modelize heterogeneous systems
- ▶ Agents' goals are not necessarily antagonistic
- ▶ Controller synthesis

 *Casting*



Formal model

Definition

$$\mathcal{G} = \langle \text{States}, \text{Agt}, \text{Act}, \text{Mov}, \text{Tab}, (\text{Allow}_A)_{A \in \text{Agt}}, (\phi_A)_{A \in \text{Agt}} \rangle$$

with

- ▶ $|\text{States}| < +\infty$, $|\text{Agt}| < +\infty$ *and* $|\text{Act}| < +\infty$
- ▶ $\text{Tab} : \text{States} \times \text{Act}^{\text{Agt}} \longrightarrow \text{States}$
- ▶ $\forall A \in \text{Agt} \quad \text{Allow}_A : \text{States} \longrightarrow 2^{\text{Act} \setminus \{\emptyset\}}$
- ▶ $\forall A \in \text{Agt} \quad \phi_A : \text{States}^\omega \longrightarrow \mathbb{R}$

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Terminal reward objectives: $\phi_A(r) \neq 0 \Leftrightarrow \exists s \in \text{States} \quad s \in \text{inf}(r) \wedge \text{Tab}(s, \Pi_B(\text{Allow}_B(s))) = \{s\}$

Formal model: strategies and outcome

Definition

A strategy σ_A for agent A is a mapping from histories to allowed actions. If $h \in \text{States}^+$ is an history

$$\sigma_A(h) \in \text{Allow}_A(\text{last}(h))$$

S_A is the set of strategies for agent A and $S = (S_A)_{A \in \text{Agt}}$ the set of strategy profiles

Definition (Semantics)

Let $h \in \text{States}^+$ an history and $\sigma \in S$

Next state is $\text{Tab}(\text{last}(h), (\sigma_A(h))_A)$. We define $h_0 = h$ and $h_{n+1} = h_n \cdot \text{Tab}(\text{last}(h_n), (\sigma_A(h_n))_A)$. The outcome of σ starting from h is defined by $\text{Out}(\sigma, h) = \lim h_n$

Pure Nash Equilibrium

Definition

Let σ a strategy profile and h an history, then (σ, h) is a Nash Equilibrium if for all agent A and any deviation $\sigma'_A \in S_A$,

$$\phi_A(\text{Out}(\sigma[A/\sigma'_A], h)) \leq \phi_A(\sigma, h)$$

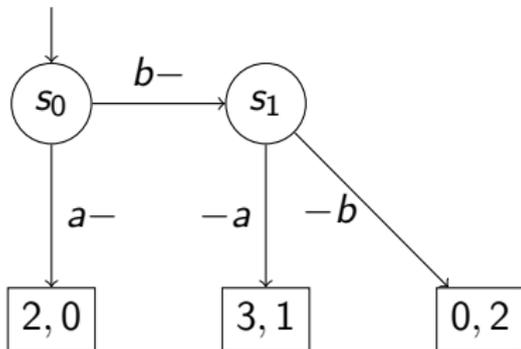
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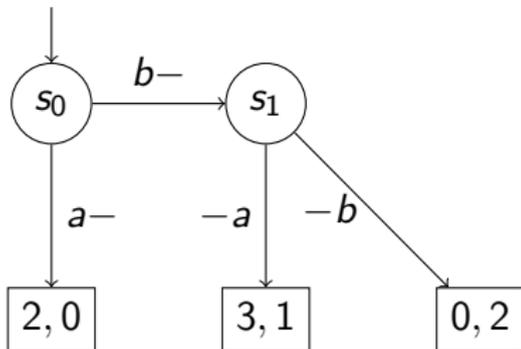


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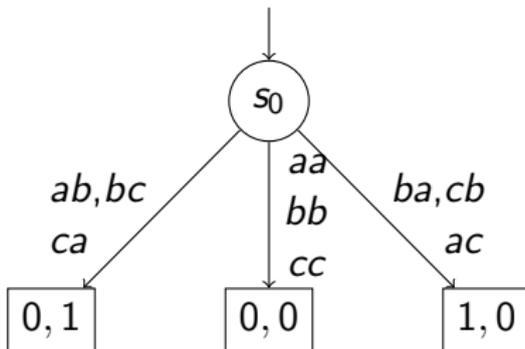
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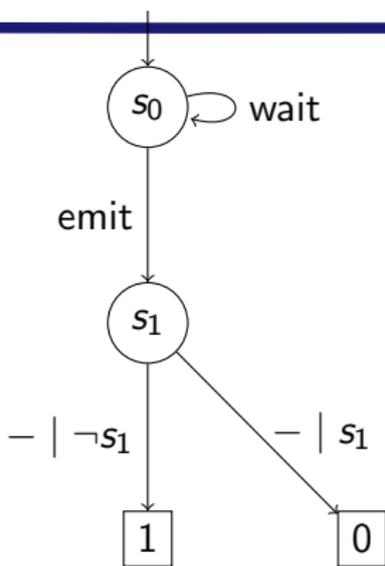
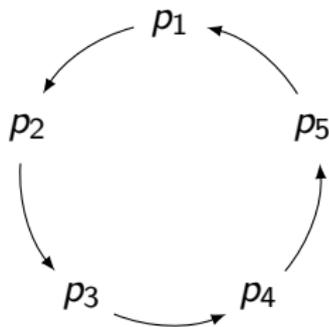
Only equilibrium value is $(2, 0)$. Note: the definition of $\sigma(h)$ is important even when h is not a prefix of $\text{Out}(\sigma, h)$ (retaliation). However, it is not clear who is the player to punish when a deviation occurs. (see Suspect game [?])

Need for mixed strategies

Main goal: break the symmetry



Another model



Question: does there exist a local strategy ensuring that state 1^n is reached ?

- ▶ Deterministically: no
- ▶ Almost surely: no
- ▶ Limit surely: yes

Formal model: strategies and outcome

Definition

A strategy σ_A for agent A is a mapping from histories to distribution of allowed actions. If $h \in \text{States}^+$ is an history

$$\sigma_A(h) \in \text{Dist}(\text{Allow}_A(\text{last}(h)))$$

\mathbb{S}_A is the set of strategies for agent A and $S = (\mathbb{S}_A)_{A \in \text{Agt}}$ the set of mixed strategy profiles.

Definition (Semantics)

$\text{Out}(\sigma, h)$ is now a random variable. For any function ϕ , denote by $\mathbb{E}^\sigma(\phi \mid h) = \mathbb{E}(\text{Out}(\sigma, h))$

mixed Nash Equilibrium

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Let σ a strategy profile and h an history, then (σ, h) is a Nash Equilibrium if for all agent A and any deviation $\sigma'_A \in S_A$,

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We can show that we can restrict to deterministic deviation only (for reachability objectives).

Does a mixed Nash Equilibrium always exist?

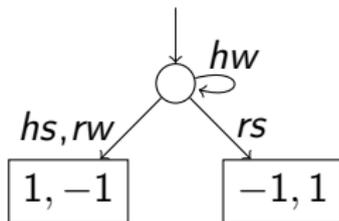
Theorem (Nash [?])

Every one-stage game has a Nash Equilibrium in mixed strategies.

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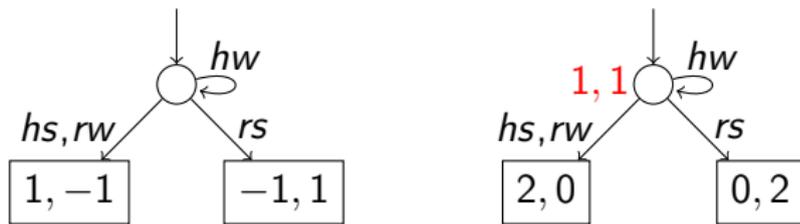


Figure: Hide-or-Run game

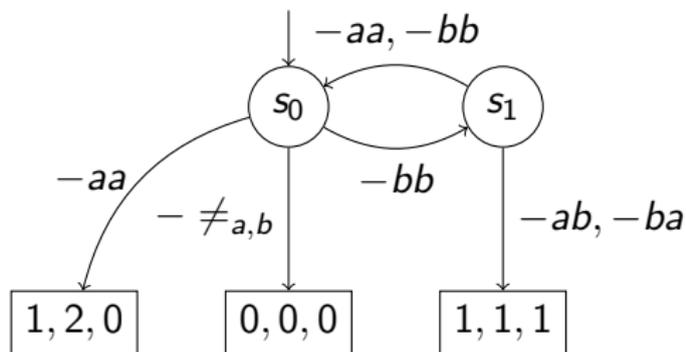
Value problem in a zero-sum game is *not* a special case of Nash Equilibria problem with positive rewards

Mixed NE can be complex

- ▶ A play is now a tree
- ▶ Irrational values may appear
- ▶ Strategies in the support must be optimal (\sim sure winning)
- ▶ Zero-Sum values have to be considered for deviation (\sim limit sure winning)

Mixed NE can be complex

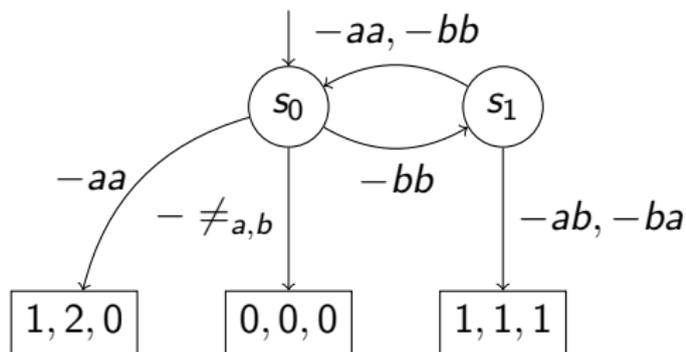
And there may be a countable number of equilibria.



$$\{v = \mathbb{E}^\sigma(\phi \mid s_0) \mid v_0 = 1 \wedge (\sigma, s_0) \text{ Nash Equilibrium}\} =$$

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And there may be a countable number of equilibria.



$$\{v = \mathbb{E}^\sigma(\phi \mid s_0) \mid v_0 = 1 \wedge (\sigma, s_0) \text{ Nash Equilibrium}\} = \\ \{(1, 1 + \frac{1}{2^k}, 1 - \frac{1}{2^k} \mid k \in \mathbb{N} \vee k = \infty\}$$

(Un)-Decidability

Theorem

The constrained existence problem is undecidable for 3-player concurrent games with terminal positive reward games.

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Sketch of the proof.

Reduction from the non-halting problem for 2-counters machine. Encodes the counters as payoff $(1, 1 + \frac{1}{2^x 3^y}, 1 - \frac{1}{2^x 3^y})$. In order to keep the same values in different branches of the game, we encode module games in a single equivalent state. \square

Overview

- ▶ Concurrent games
- ▶ Mixed strategies are useful
- ▶ Rich framework
- ▶ ...both for equilibria or simple reachability

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- ▶ Concurrent games
- ▶ Mixed strategies are useful
- ▶ Rich framework
- ▶ ...both for equilibria or simple reachability

- ▶ Still hope for the 2 agent case
- ▶ May be extended to $n - \frac{1}{2}$ -games
- ▶ Extension to more players but with more structure

Thank you for your attention

Bibliography I
