Refinements for free![1]

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Motivation

Verifying computer algebra algorithms

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Verifying computer algebra algorithms

What for?

- Computer algebra algorithms can help automate proofs
- Formal proofs bridge the gap between paper correctness proofs and real-life implementations
- Proof assistants can provide independent verification of results obtained by computer algebra programs (e.g. $\zeta(3)$ is irrational, computation of homology groups)

Context

Traditional approaches to program verification:

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- Bottom-up verification (e.g. annotations)

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- Computer algebra algorithms can have complex specifications
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Specificity of computer algebra programs:

- Computer algebra algorithms can have complex specifications
- Efficiency matters!

Problem: these aspects are often in tension
We suggest a methodology based on refinements to achieve separation of concerns

Separation of concerns

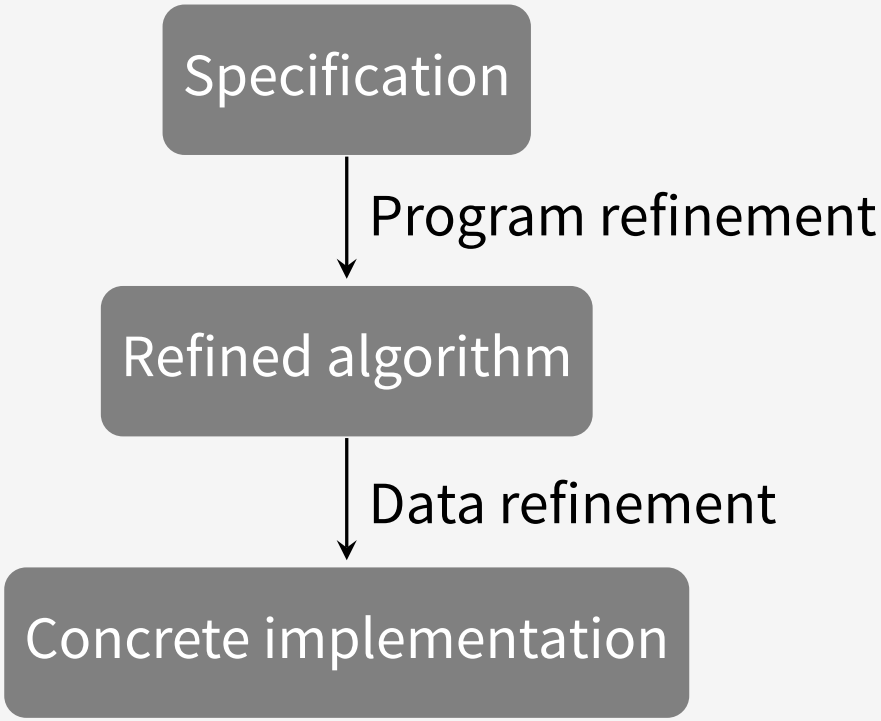
We know that a program must be correct and we can study it from that viewpoint only; we also know that it should be efficient and we can study its efficiency on another day, so to speak. […] But nothing is gained – on the contrary! – by tackling these various aspects simultaneously. It is what I sometimes have called "the separation of concerns"

Dijkstra, Edsger W.
"On the role of scientific thought" (1982)

Program and data refinements

We distinguish two kinds of refinements:

- Program refinement: improving the algorithmics
- Data refinement: switching to more efficient data representation



Isomorphic structures

First example: natural numbers

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Partial operators

Second example: polynomials in SSReflect

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Variable R : ringType.
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Operators over `(seq R)` are partially specified as refinements of their counterparts from `(polynomial R)`.

Quotient

Third example: rational numbers

```
Record rat : Set := Rat {
  valq : (int * int) ;
  _ : (0 < valq.2) && coprime `|valq.1| `|valq.2|
}.
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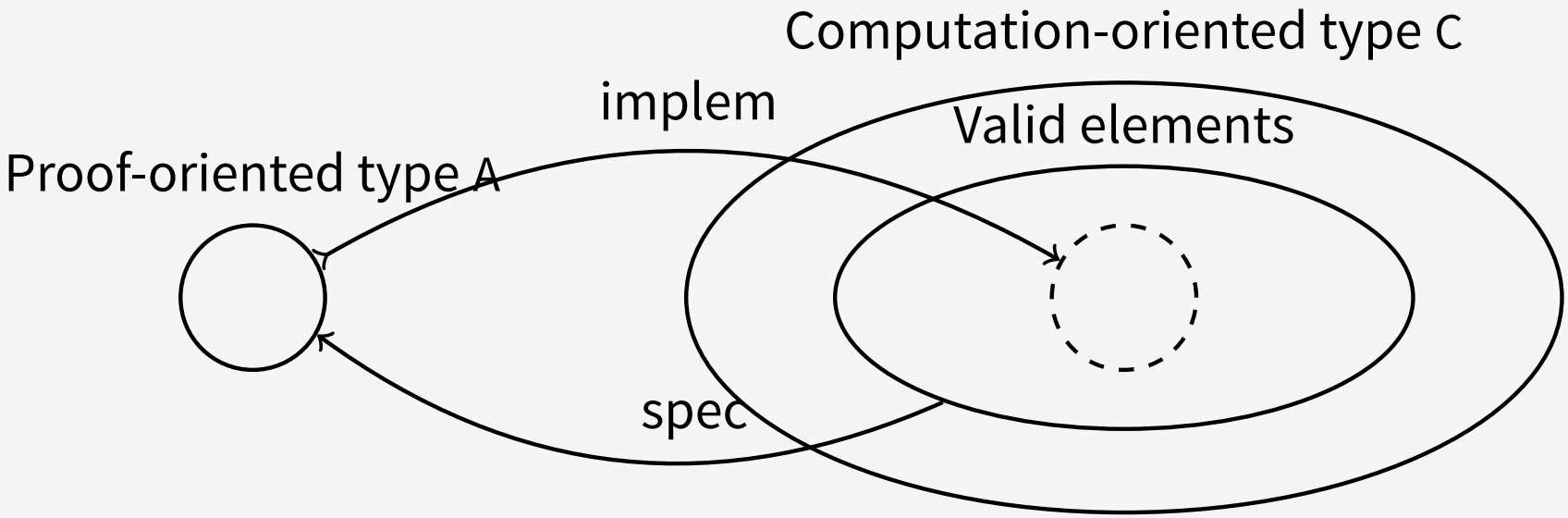
The proof-oriented rat enforces that fractions are reduced

- Allows to use Leibniz equality in proofs
- This invariant is costly to maintain during computations

We would like to relax the constraint and express that `rat` is isomorphic to a quotient of a subset of pairs of integers.

Representation of refinements



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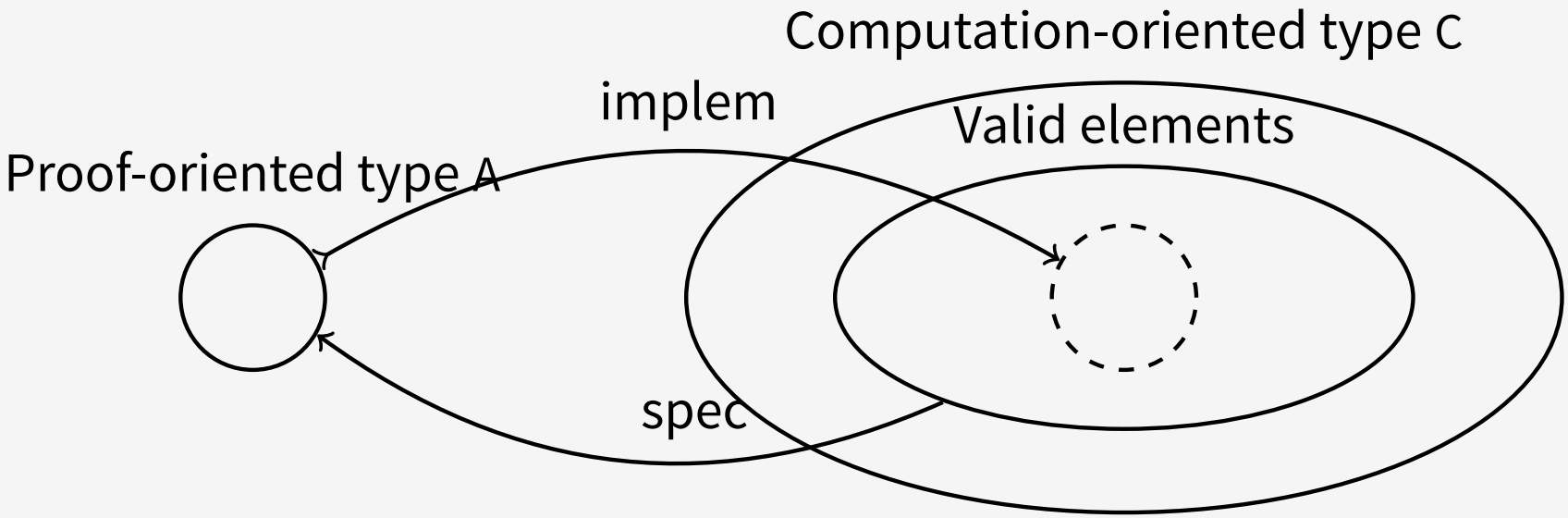

The old strategy

Assuming we have a theory on a type `A`:

1. write efficient algorithms for `A`,
2. prove that `A` and `C` are isomorphic,
3. duplicate the algorithms for `C`,
4. prove extensional equality of algorithms.

The new strategy

Assuming we have a theory on a type A:

1. write efficient algorithms in a generic form,
2. instantiate in A and prove them,
3. instantiate in C and get correction by parametricity.

New features

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- Compositionality: refining `(polynomial A)` to `(seq C)`.
- Automating correctness proofs when changing representations.

Generic programming: addition over rationals

Generic datatype

```
Definition Q Z := (Z * Z).
```

Generic operations

```
Definition addQ Z `{add Z} `{mul Z} : add (Q Z) :=
  fun x y => (x.1 * y.2 + y.1 * x.2, x.2 * y.2).
```

To prove correctness of addQ, operators `(+ : add Z)` and `(* : mul Z)` are instanciated to proof-oriented definitions.
When computing, these operators are instanciated to more efficient ones.

Proof-oriented correction

- The type `int` is the proof-oriented version of integers.
- The type `rat` is the proof-oriented version of rationals.

Correctness of `addQ int`

```
Definition addQ Z `{add Z} `{mul Z} : add (Q Z) :=
  fun x y => (x.1 * y.2 + y.1 * x.2, x.2 * y.2).

Definition Rrat : rat -> Q int -> Prop := ofun_hrel
    Qint_to_rat.

Lemma Rrat_addQ :
  forall (x : rat) (u : Q int), Rrat x u ->
  forall (y : rat) (v : Q int), Rrat y v ->
  Rrat (addq x y) (addQ u v).
```

Compositionality

Composing relations

```
Definition comp_hrel
 (R : A -> B -> Prop) (R' : B -> C -> Prop) : A -> C ->
      Prop :=
  fun a c => exists b, R a b /\ R' b c.
Notation "X \o Y" := (comp_hrel X Y).
```

Example for `rat`

```
Definition Rrat : rat -> Q int -> Prop := ofun_hrel
    Qint_to_rat.

Variables (Z : Type) (Rint : int -> Z -> Prop).
Definition RratA : rat -> Q Z -> Prop :=
  (Rrat \o (Rint * Rint))%rel.
```

Compositionality

Correctness of addQ

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Definition addQ Z `{add Z} `{mul Z} : add (Q Z) :=
  fun x y => (x.1 * y.2 + y.1 * x.2, x.2 * y.2).

Variables (Z : Type) (Rint : int -> Z -> Prop).
Definition RratA := (Rrat \o (Rint * Rint))%rel.

Lemma RratA_addQ `{add Z, mul Z} : [...] ->
  forall (x : rat) (u : Q Z), RratA x u ->
  forall (y : rat) (v : Q Z), RratA y v ->
  RratA (addq x y) (addQ u v).
```

This will be provable as soon as the addition and multiplication over Z refines the ones over `int`.

Automation

Correctness of addQ

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Definition addQ Z `{add Z} `{mul Z} : add (Q Z) :=
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```

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  (Rint ==> Rint ==> Rint) mulz (*) ->
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    addz mulz)

Lemma param_addQ `{add Z, mul Z} :
  (Rint ==> Rint ==> Rint) addz (+) ->
  (Rint ==> Rint ==> Rint) mulz (*) ->
  (Rint * Rint ==> Rint * Rint ==> Rint * Rint)
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- `Rrat_addQ` is not for free,

Automation

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  (Rint * Rint ==> Rint * Rint ==> Rint * Rint)
    (addQ addz mulz) (addQ (+) (*))
```

- `Rrat_addQ` is not for free,
- but `param_addQ` should be!

Parametricity for addQ

```
  Z : Type
  Rint : int -> Z -> Prop
  addZ : add Z
  mulZ : mul Z
  _ : (Rint ==> Rint ==> Rint) addz (+)
  _ : (Rint ==> Rint ==> Rint) mulz (*)
============================================================
(RratA ==> RratA ==> RratA) addq (@addQ Z (+) (*))
```

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============================================
(Rrat ==> Rrat ==> Rrat)
 addq (@addQ int addz mulz)


============================================
(Rint * Rint ==> Rint * Rint ==> Rint * Rint)
 (@addQ int addz mulz) (@addQ Z (+) (*))
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(Rint * Rint ==> Rint * Rint ==> Rint * Rint)
  (@addQ int addz mulz) (@addQ Z (+) (*))
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((Rint ==> Rint ==> Rint) ==>
      Rint * Rint ==> Rint * Rint ==> Rint * Rint)
 (@addQ int addz) (@addQ Z (+))
```

Parametricity for addQ

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  Z : Type
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((Rint ==> Rint ==> Rint) ==>
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      Rint * Rint ==> Rint * Rint ==> Rint * Rint)
 (@addQ int) (@addQ Z)
```

Conclusion and future work

The approach we described:

- Reconciles convenient proofs with efficient computations
- Provides a mechanism to smoothly switch from one world to the other
- Avoids duplication of code

We

- applied it to algorithms we had previously verified: Karatsuba's polynomial multiplication, Strassen's matrix product,
- are still porting others from the old framework: Sasaki-Murao algorithm, Smith normal form.

Future work:

- have a better way to get parametricity than typeclasses,
- try on algorithms outside algebra,
- scale up to dependant types.

Thanks!