

# The Čech complex

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$$0 \longrightarrow \check{C}_{\underline{a}}^0(\mathbf{A}) \longrightarrow \check{C}_{\underline{a}}^1(\mathbf{A}) \longrightarrow \dots \longrightarrow \check{C}_{\underline{a}}^n(\mathbf{A}) \longrightarrow 0$$

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and

$$d : \bigoplus_{\#I=k} \mathbf{A}_{a_I} \longrightarrow \bigoplus_{\#J=k+1} \mathbf{A}_{a_J}, \quad x_I \mapsto \bigoplus_{i \notin I} (-1)^{\varepsilon_i(I)} (x_I)_{I \cup i}$$

where  $\varepsilon_i(I)$  is the number of elements of  $I$  less than  $i$ .

# The Čech complex

For  $n = 3$

$$\begin{aligned} \mathbf{A} &\longrightarrow \mathbf{A}_{a_1} \oplus \mathbf{A}_{a_2} \oplus \mathbf{A}_{a_3} \longrightarrow \mathbf{A}_{a_1 a_2} \oplus \mathbf{A}_{a_1 a_3} \oplus \mathbf{A}_{a_2 a_3} \longrightarrow \mathbf{A}_{a_1 a_2 a_3} \\ x &\longmapsto (x, x, x) \\ (x_1, x_2, x_3) &\longmapsto (x_2 - x_1, x_3 - x_1, x_3 - x_2) \\ (x_{12}, x_{13}, x_{23}) &\longmapsto x_{23} - x_{13} + x_{12} \end{aligned}$$

The Čech complex measures, like the Koszul complex, the depth of  $\langle \underline{a} \rangle$ . In other words

$$\text{Gr}(\underline{a}; \mathbf{A}) \geq k \Leftrightarrow \check{H}_{\underline{a}}^i(\mathbf{A}) = 0, \forall i < k$$

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$$H^i(\underline{a}; \mathbf{A}) = 0, \forall i < k \Leftrightarrow \check{H}_{\underline{a}}^i(\mathbf{A}) = 0, \forall i < k$$

# What is a bicomplex ?

Let  $(K^{p,q})_{p,q \in \mathbb{Z} \times \mathbb{Z}}$  be a collection of modules with horizontal and vertical morphisms  $d' : K^{p,\bullet} \rightarrow K^{p+1,\bullet}$  and  $d'' : K^{\bullet,q} \rightarrow K^{\bullet,q+1}$  such that  $d' \circ d' = 0$ ,  $d'' \circ d'' = 0$  and  $d' \circ d'' = d'' \circ d'$ .

We say that

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is a **bicomplex**. To this bicomplex is associated a **total complex**  $(T^\bullet, d)$  with

$$T^n = \bigoplus_{p+q=n} K^{p,q}$$

and

$$\forall a_{p,q} \in K^{p,q}, \quad d(a_{p,q}) = d'(a_{p,q}) + d''(a_{p,q})$$



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In the following results, the hypothesis focuses on

row-exactness or/and column-exactness

on one (or two) diagonal(s).

Rmk: the vertical direction (i.e. column-exactness) is favored.

The conclusion focuses on the cohomology of the total complex, i.e.  $H^n(T^\bullet) := \text{Ker } d_n / \text{Im } d_{n+1}$ .

From cohomological informations on the bicomplex (in a given direction: row or column), we can obtain information about the cohomology of the total complex.

## Le lemme de l'escalier

Let  $n \in \mathbb{N}$ . If there is column-exactness on the diagonal  $n$ , then the total complex is exact in degree  $n$ . In other words:

$$\forall p + q = n, H^q(K^{p, \bullet}) = 0 \implies H^n(T^\bullet) = 0$$

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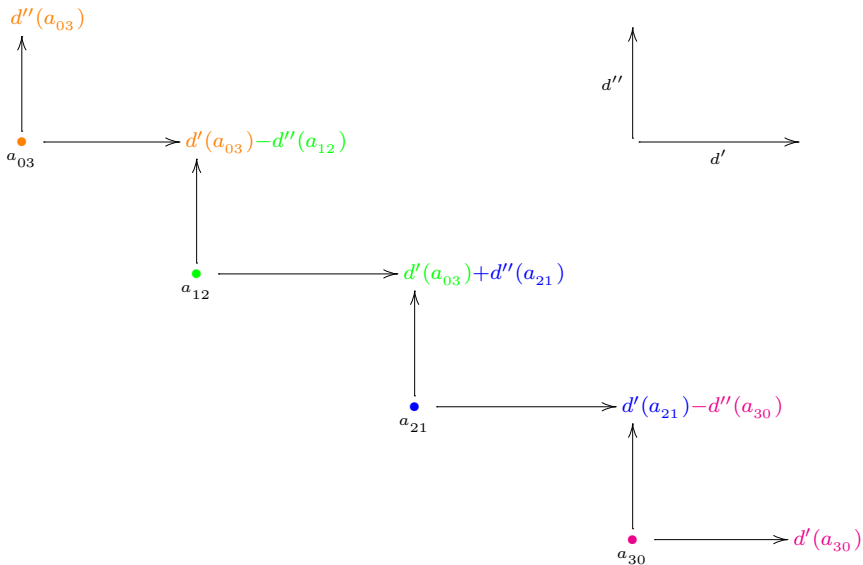
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For the proof, let's take  $n = 3$ . Let  $a = a_{03} \oplus a_{12} \oplus a_{21} \oplus a_{30} \in T^3$  such that  $d(a) = 0$ . With the hypothesis of column-exactness, we want to find  $b \in T^2$  such that  $a = d(b)$ .

# Diagram chasing



# Diagram chasing

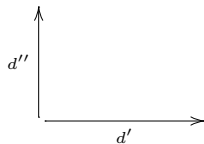
$$d''(a_{03}) = 0$$

$$a_{03} \longrightarrow d'(a_{03}) - d''(a_{12}) = 0$$

$$a_{12} \longrightarrow d'(a_{03}) + d''(a_{21}) = 0$$

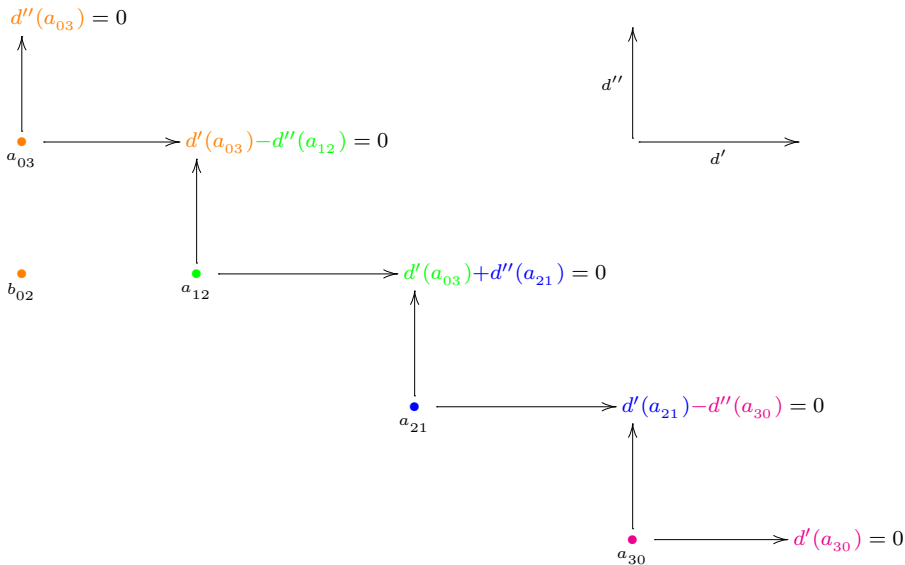
$$a_{21} \longrightarrow d'(a_{21}) - d''(a_{30}) = 0$$

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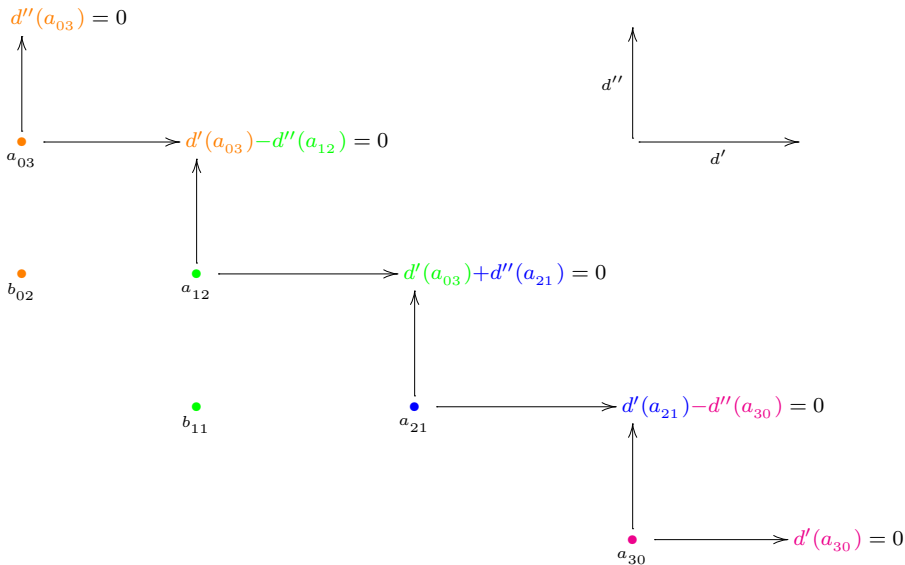




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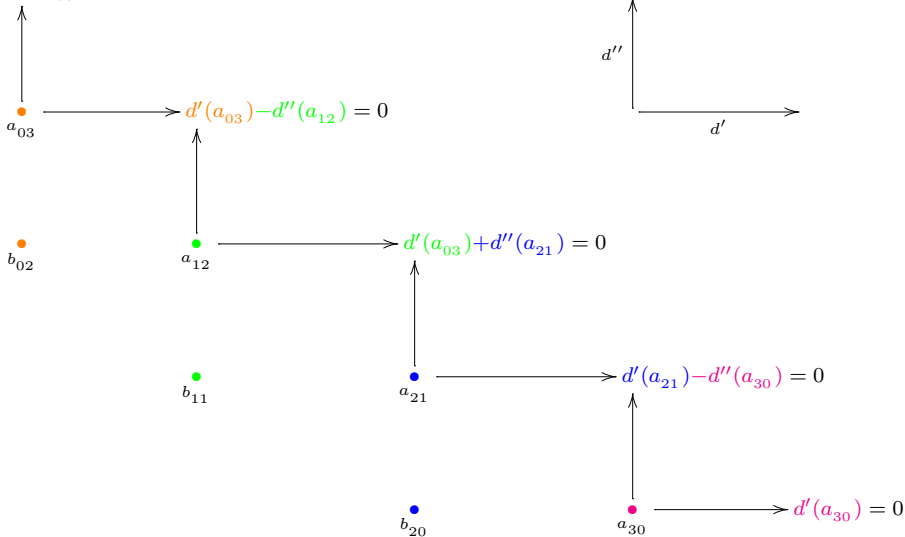


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$$d''(a_{03}) = 0$$



## Column

If there is column-exactness on the diagonals  $n$  and  $n + 1$  except perhaps on the  $(Oy)$  axis, then  $H^n(K^{\bullet,0}) \simeq H^n(T^\bullet)$ .

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### Row

If there is row-exactness on the diagonals  $n$  and  $n + 1$  except perhaps on the  $(Ox)$  axis, then  $H^n(K^{0,\bullet}) \simeq H^n(T^\bullet)$ .

# The $(Ox)$ -complex $(K^{0,\bullet}, d')$ , The $(Oy)$ -complex $(K^{\bullet,0}, d'')$

## Column

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## Column and Row

If there are column-exactness and row-exactness on the diagonals  $n$  and  $n + 1$  except perhaps on the  $(Ox)$  and  $(Oy)$  axis, then  $H^n(K^{0,\bullet}) \simeq H^n(T^\bullet) \simeq H^n(K^{\bullet,0})$ .

Čech-depth  $\Leftrightarrow$  Koszul-depth

$$\check{H}_{\underline{a}}^i(\mathbf{A}) = 0, \forall i < k \iff H^i(\underline{a}; \mathbf{A}) = 0, \forall i < k$$

We prove  $\Rightarrow$ .





We introduce the bicomplex  $K^p(\underline{a}; \mathbf{A}) \otimes \check{C}_{\underline{a}}^q(\mathbf{A})$

$$\begin{array}{ccccccc}
 & & \check{C}_{\underline{a}}^{\bullet}(\mathbf{A}) & & & & \\
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 \longrightarrow & K^0 \otimes \check{C}^2 & \longrightarrow & K^1 \otimes \check{C}^2 & \longrightarrow & K^2 \otimes \check{C}^2 & \longrightarrow & K^3 \otimes \check{C}^2 & \longrightarrow & \dots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
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 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & 0 & & 0 & & 0 & & 0 & & 
 \end{array}$$

Row direction: If there is row-exactness on the diagonals  $i$  and  $i + 1$  except perhaps on the  $(Ox)$  axis, then  $H^i(a; \mathbf{A}) \simeq H^i(T^{\bullet})$ .

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Column direction: If there is column-exactness on the diagonal  $i$ , then  $H^i(T^{\bullet}) = 0$ .

## Column

If there is column-exactness on the diagonals  $n$  and  $n + 1$  except perhaps on the  $(Ox)$  axis, then  $H^n(X^\bullet) \simeq H^n(T^\bullet)$  where

$$X^p = \text{Ker} \begin{pmatrix} K^{p,1} \\ \uparrow \\ K^{p,0} \end{pmatrix}$$

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## Row

If there is row-exactness on the diagonals  $n$  and  $n + 1$  except perhaps on the  $(Oy)$  axis, then  $H^n(Y^\bullet) \simeq H^n(T^\bullet)$  where

$$Y^q = \text{Ker} (K^{q,0} \rightarrow K^{q,1})$$

# The $X_\bullet$ complex and $Y_\bullet$ complex

## Column and Row

If there are column-exactness and row-exactness on the diagonals  $n$  and  $n - 1$  except perhaps on the  $(Ox)$  and  $(Oy)$  axis, then  $H_n(X_\bullet) \simeq \quad \simeq H_n(Y_\bullet)$  where

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# The Tor foundation

Let  $M$  be a  $\mathbf{A}$ -module and  $P_\bullet$  a projective resolution of  $M$

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow\!\!\!\twoheadrightarrow M$$

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$$X_\bullet = P_\bullet \otimes N \quad \text{and} \quad Y_\bullet = M \otimes Q_\bullet$$

The previous proposition says :

$$H_i(P_\bullet \otimes N) \simeq \quad \simeq H_i(M \otimes Q_\bullet)$$



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The previous proposition says :

$$H_i(P_\bullet \otimes N) \simeq \text{Tor}_i^{\mathbf{A}}(M, N) \simeq H_i(M \otimes Q_\bullet)$$

# The Hochster-formula

Let  $\mathfrak{a}$  be a *monomial* ideal of  $\mathbf{S} = \mathbf{k}[X_1, \dots, X_n]$ . Consider the graded (for fine grading) minimal free resolution of  $\mathbf{S}/\mathfrak{a}$

$$0 \longrightarrow L_n \longrightarrow \cdots \longrightarrow L_1 \longrightarrow L_0 \twoheadrightarrow \mathbf{S}/\mathfrak{a}$$

with

$$L_i = \bigoplus_{\alpha \in \mathbb{N}^n} \mathbf{S}(-\alpha)^{\beta_{i,\alpha}}$$

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The Hochster's formula computes the multigraded Betti numbers of  $\mathbf{S}/\mathfrak{a}$  via simplicial cohomology.

## Hochster's formula

$$\beta_{i,\alpha} = \dim_{\mathbf{k}} \tilde{H}^{|\text{supp}(\alpha)|-i-1}(\mathcal{K}_\alpha(\mathfrak{a}); \mathbf{k})$$

where  $\mathcal{K}_\alpha(\mathfrak{a}) = \{ J \subset \text{supp}(\alpha) \mid X^{\alpha - \text{supp}(\alpha)} X^J \notin \mathfrak{a} \}$

The graded minimal free resolution of  $\mathbf{S}/\mathfrak{a}$ :

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The Koszul complex of  $\underline{X} = (X_1, \dots, X_n)$ :

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The previous result says that homology of this two complexes are isomorphic.

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- As  $L_\bullet$  is minimal, the homology of  $L_\bullet/\underline{X}L_\bullet$  is reduced to its terms. Thus  $H_i(L_\bullet/\underline{X}L_\bullet)_\alpha = (L_i/\underline{X}L_i)_\alpha = \mathbf{k}^{\beta_{i,\alpha}}$ .

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- What is the component of degree  $\alpha \in \mathbb{N}^n$  of the complex  $K_\bullet/\mathfrak{a}K_\bullet = K_\bullet(\underline{X}; \mathbf{S}/\mathfrak{a})$ ? We have

$$(K_i/\mathfrak{a}K_i)_\alpha = \bigoplus_{\#I=i} \mathbf{k} \frac{X^\alpha}{X^I} e_I$$

where  $I \subset \text{supp}(\alpha)$  is such that  $\frac{X^\alpha}{X^I} \notin \mathfrak{a}$