

Towards  
an  
Algebraic Framework  
for  
Nonlinear Dynamics

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# OUTLINE OF LECTURES

I. WHAT DO WE MEAN BY 'SOLVING' A DIFFERENTIAL EQUATION?

II. DECOMPOSING DYNAMICS

III. RECONSTRUCTING DYNAMICS

# A SUBJECTIVE HISTORICAL PERSPECTIVE OF DYNAMICS

IN THE BEGINNING:

Newton, Two body problem  $\approx$  1680

Initial Value Problem:

$$\frac{dx}{dt} = f(x, \lambda), \quad x(0) = x_0 \in \mathbb{R}^n \quad f \text{ is an explicit function}$$

$\lambda$  is a fixed parameter value

Solution:  $x: \mathbb{R} \rightarrow \mathbb{R}^n$ ,  $x(0) = x_0$   $x$  is an explicit function

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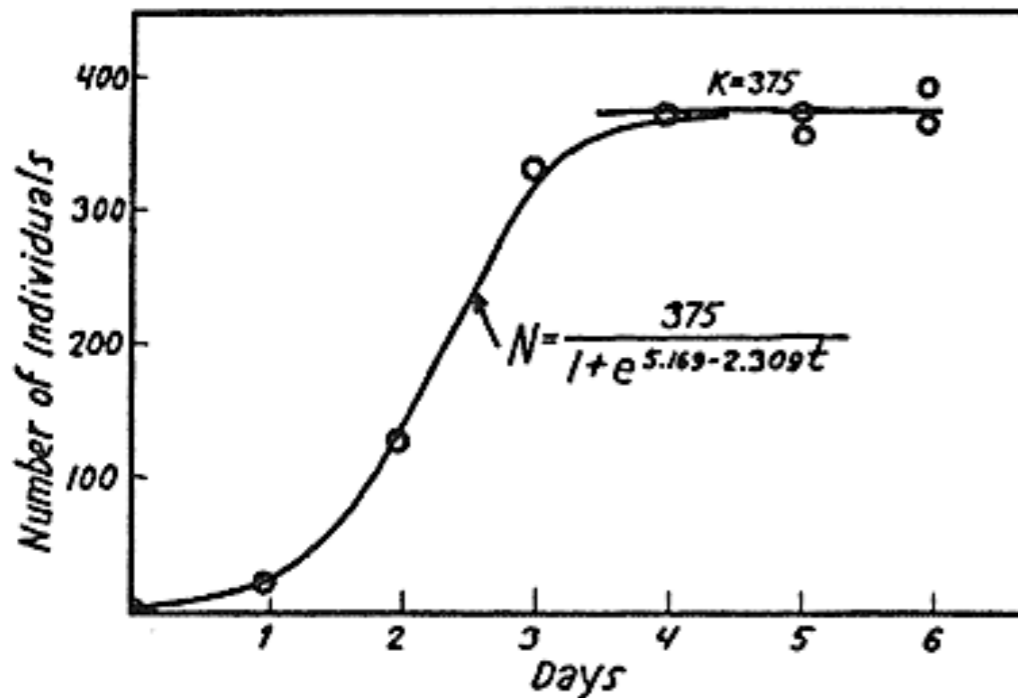
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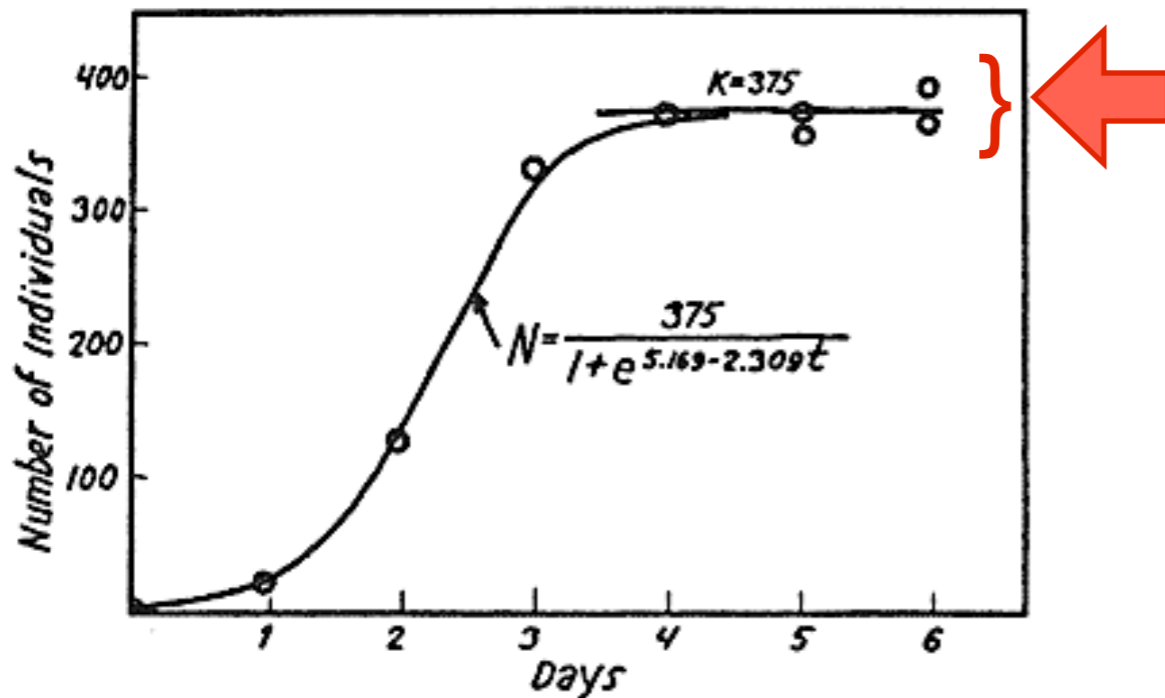
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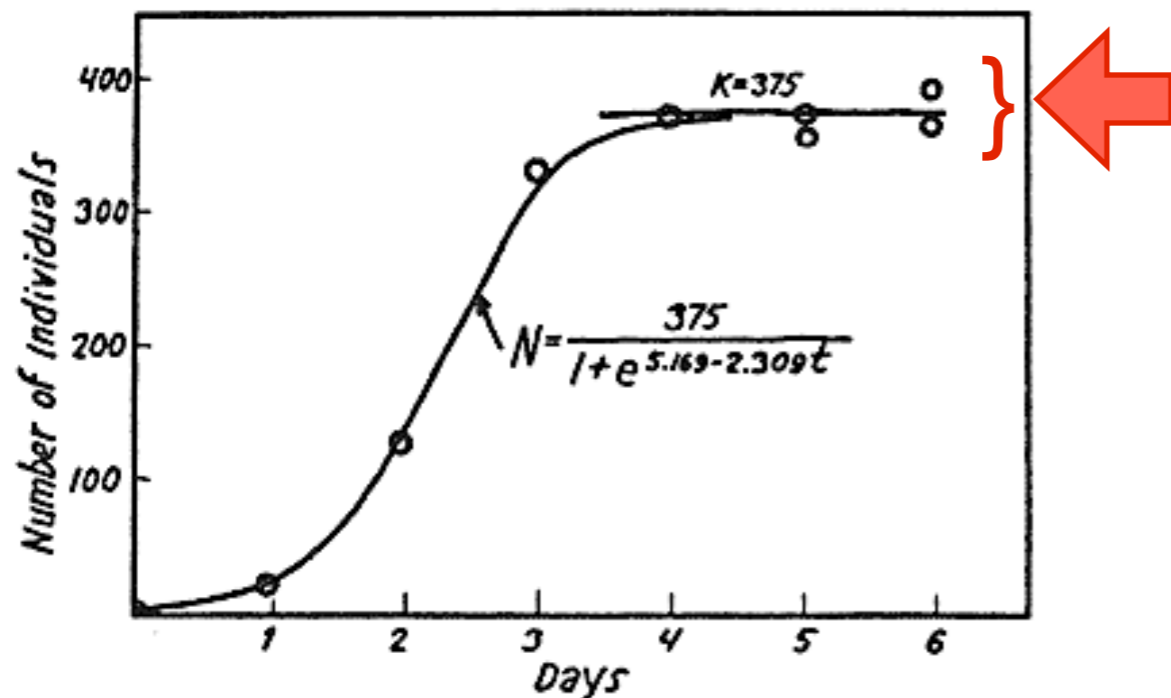
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What do we mean by  $\alpha$  and  $K$ ? Can we measure them?

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What is the maximal number of isolated periodic orbits that can occur for a particular choice of parameters?

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**Moral:** Explicit Nonlinearities are not necessarily helpful

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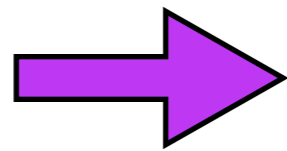
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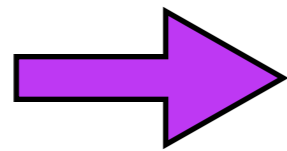


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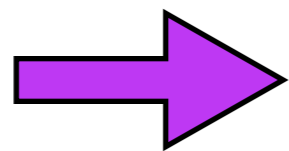
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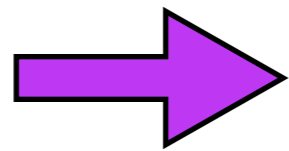
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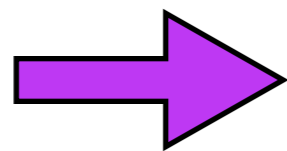
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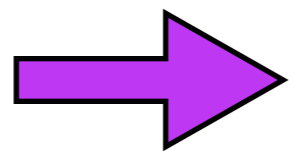


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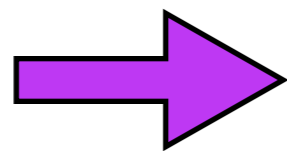
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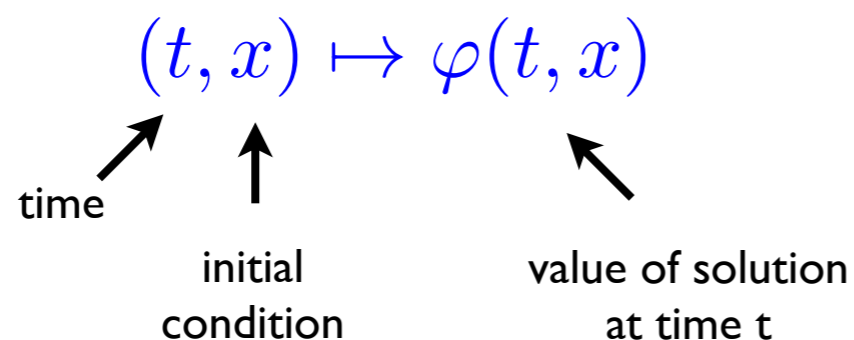


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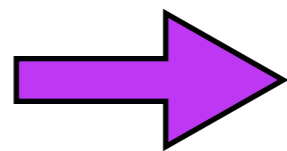
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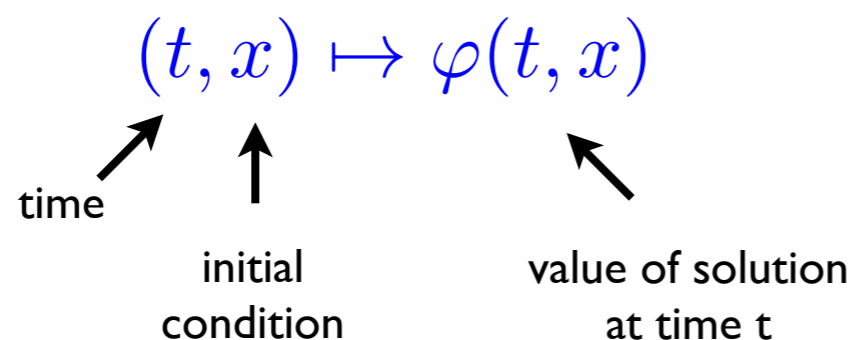
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Map:  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$

$x \mapsto f(x) := \varphi(\tau, x)$

$\tau > 0$  is a fixed time.

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The places of change

$\lambda_0 \in \Lambda$  is a **bifurcation point** if for any neighborhood  $U$  of  $\lambda_0$  there exists  $\lambda_1 \in U$  such that  $f_{\lambda_0}$  is not conjugate to  $f_{\lambda_1}$

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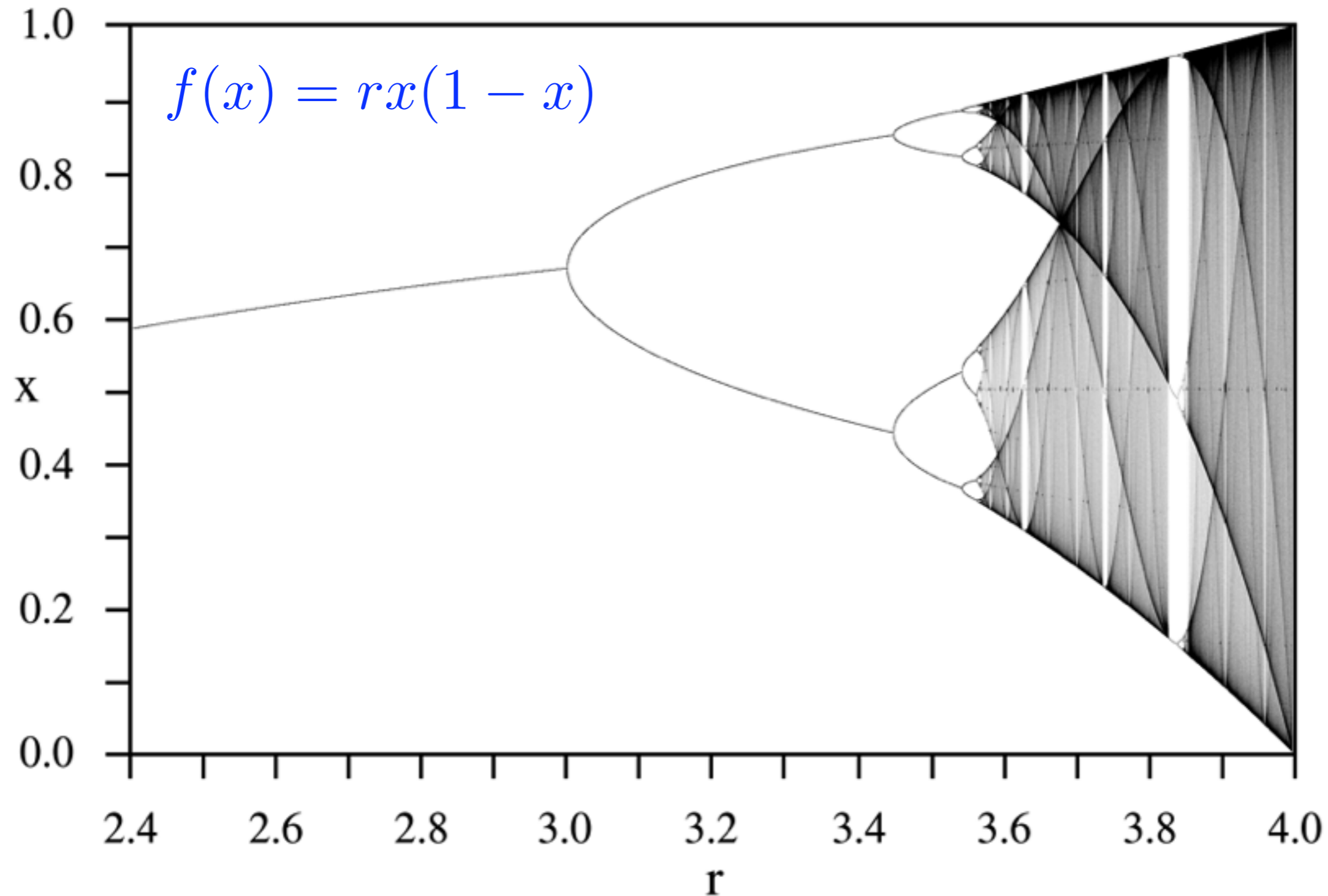
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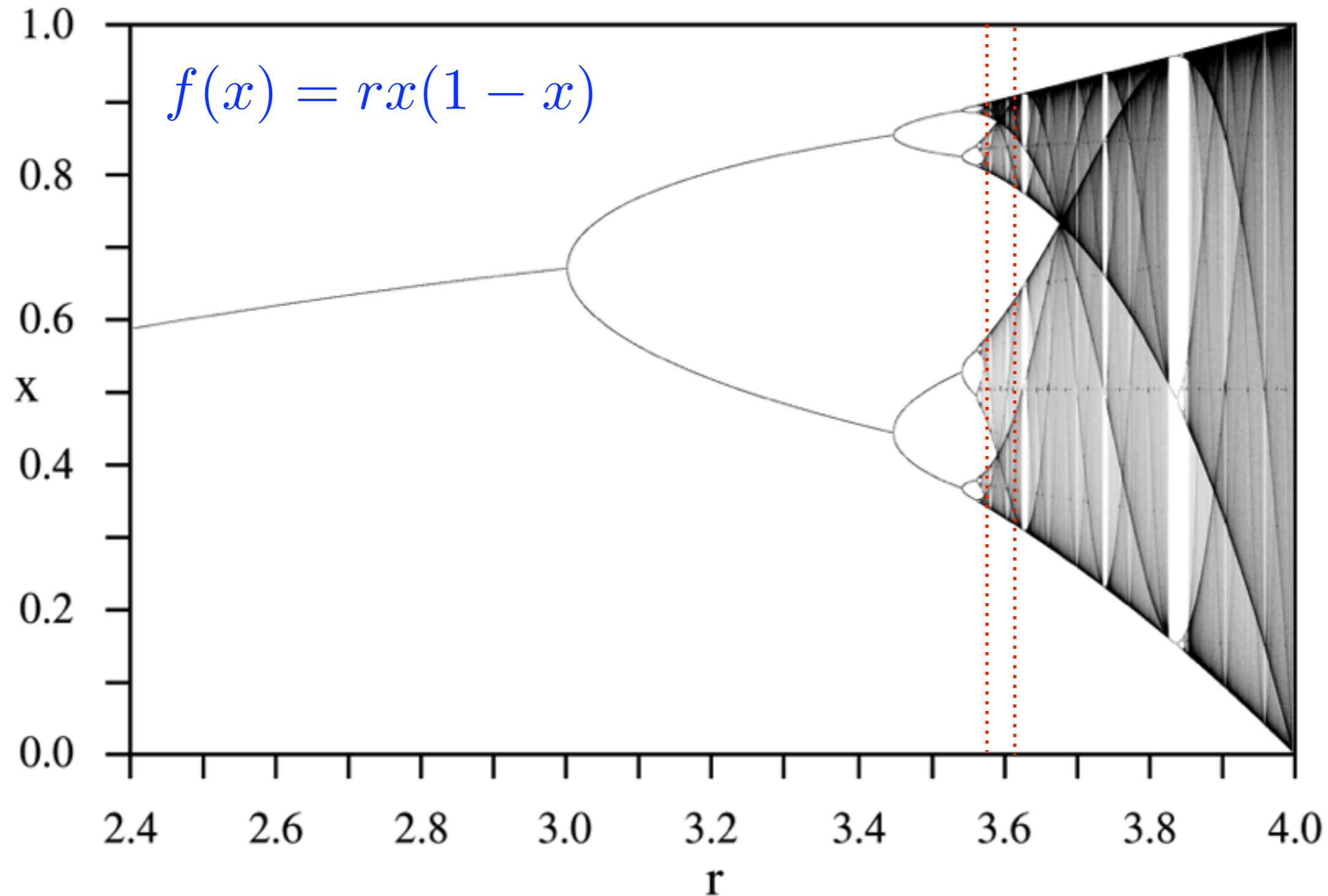
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1. We can collect enormous quantities of data. Scientists are relying more on data sets to characterize phenomena as opposed to mathematical models.
2. We can perform large scale numerical studies of nonlinear systems.
3. We are studying multiscale systems (biology, economic, medicine, etc) for which nonlinearities and parameters are not known.

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Develop a combinatorial/algebraic topological theory of nonlinear dynamics that is

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## REMARKS:

1. There are too many invariant sets so we requires a new concept for `solving' differential equations/dynamical systems.
2. Should be compatible with Classical Theory, i.e. we can relate back to invariant sets.

## II. DECOMPOSING DYNAMICS

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Recurrent vs. Nonrecurrent Dynamics

## CLASSICAL CONCEPTS

$f: X \rightarrow X$  continuous function on a compact metric space.

$\text{Invset}(f) := \{S \subset X \mid f^n(S) = S \ \forall n \in \mathbb{Z}^+\}$  invariant sets

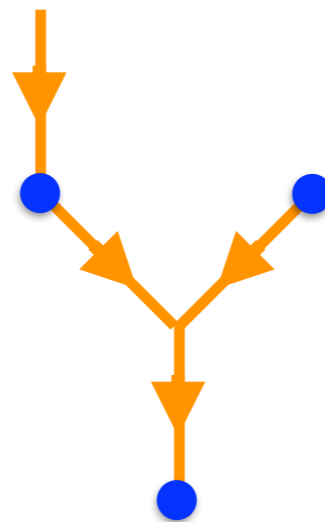
$\text{SInvset}(f) := \{S \subset X \mid f^n(S) = S \ \forall n \in \mathbb{Z}\}$  strong invariant sets

$\text{Invset}^+(f) := \{S \subset X \mid f^n(S) \subset S \ \forall n \in \mathbb{Z}^+\}$  forward invariant sets

$\text{Invset}^-(f) := \{S \subset X \mid f^n(S) \subset S \ \forall n \in \mathbb{Z}^-\}$  backward invariant sets

$\text{Invset}^\pm(f) := \text{Invset}^+(f) \cap \text{Invset}^-(f)$

Because  $f$  is not assumed to be invertible, these can all be different



# CONLEY'S DECOMPOSITION THEOREM

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Let  $\epsilon > 0$ .  $x \in X$  is  $\epsilon$ -*chain recurrent* if there exists a sequence of points  $\{x_0, x_1, \dots, x_n\} \subset X$  such that  $x_0 = x_n = x$  and  $x_{k+1} \in B_\epsilon(f(x_k))$ .

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The *chain recurrent set* is

$$\mathcal{R}(X, f) := \bigcap_{\epsilon > 0} \mathcal{R}_\epsilon(X, f).$$

**Theorem:** (Conley) Let  $X \in \text{Invset}(f)$  be compact. Then

1.  $\mathcal{R}(X, f) = \mathcal{R}(\mathcal{R}(X, f), f)$ .

2. There exists a continuous function  $V: X \rightarrow [0, 1]$  such that

(a)  $V(f(x)) \leq V(x)$  for all  $x \in X$ .

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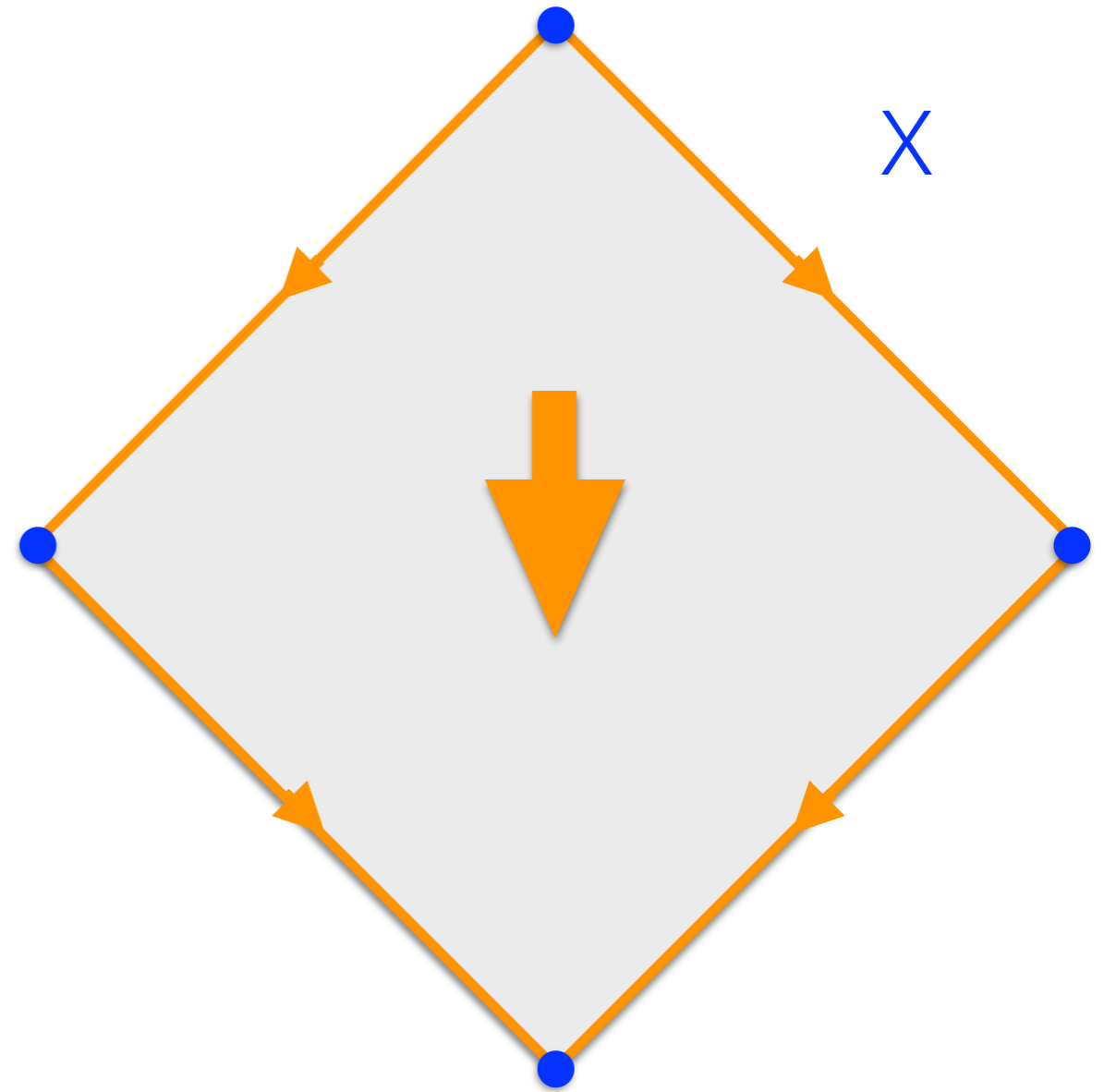
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**Goal:** Describe a constructive proof of this theorem.

W. Kalies, K. M., R. Vandervorst

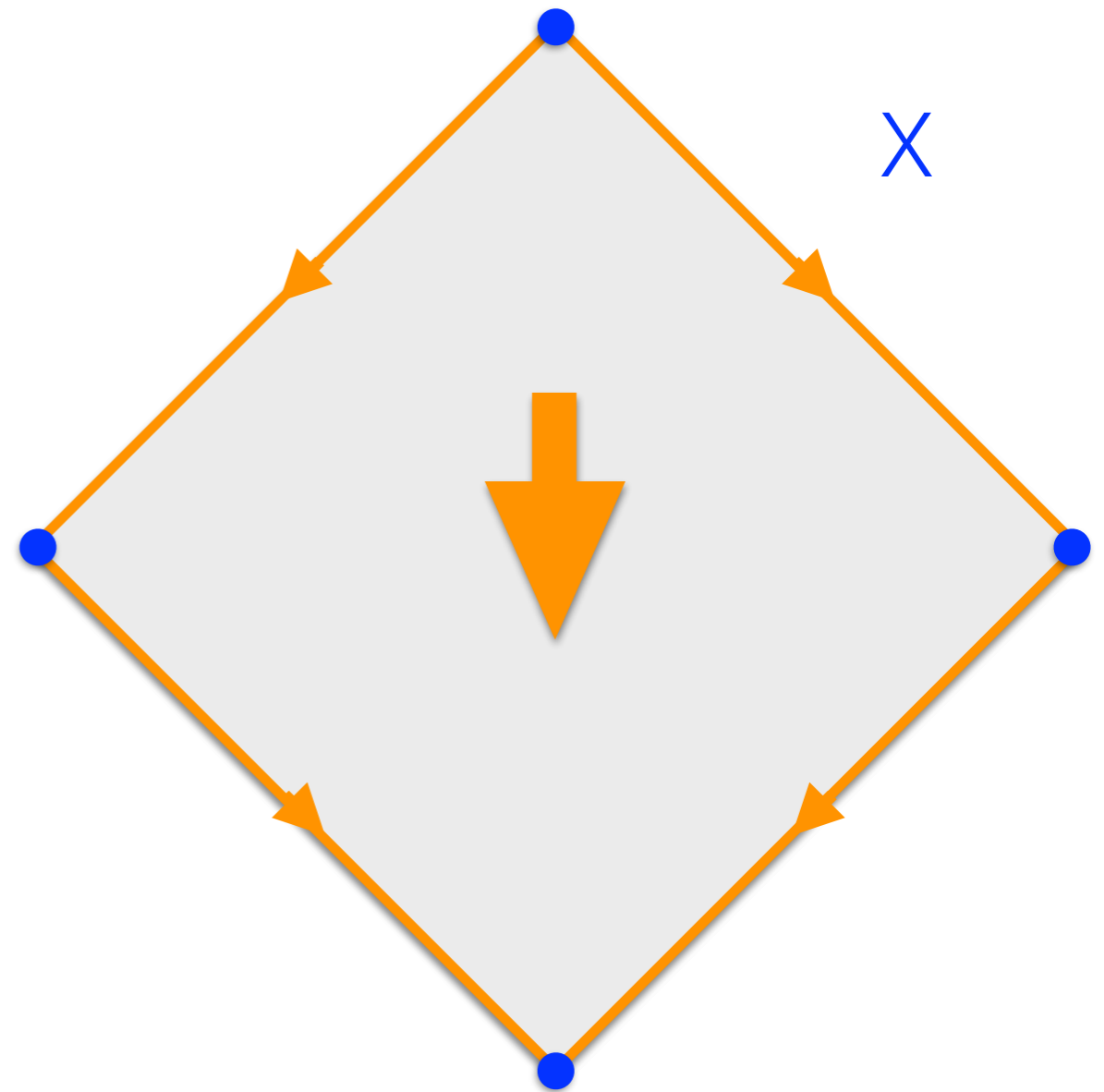
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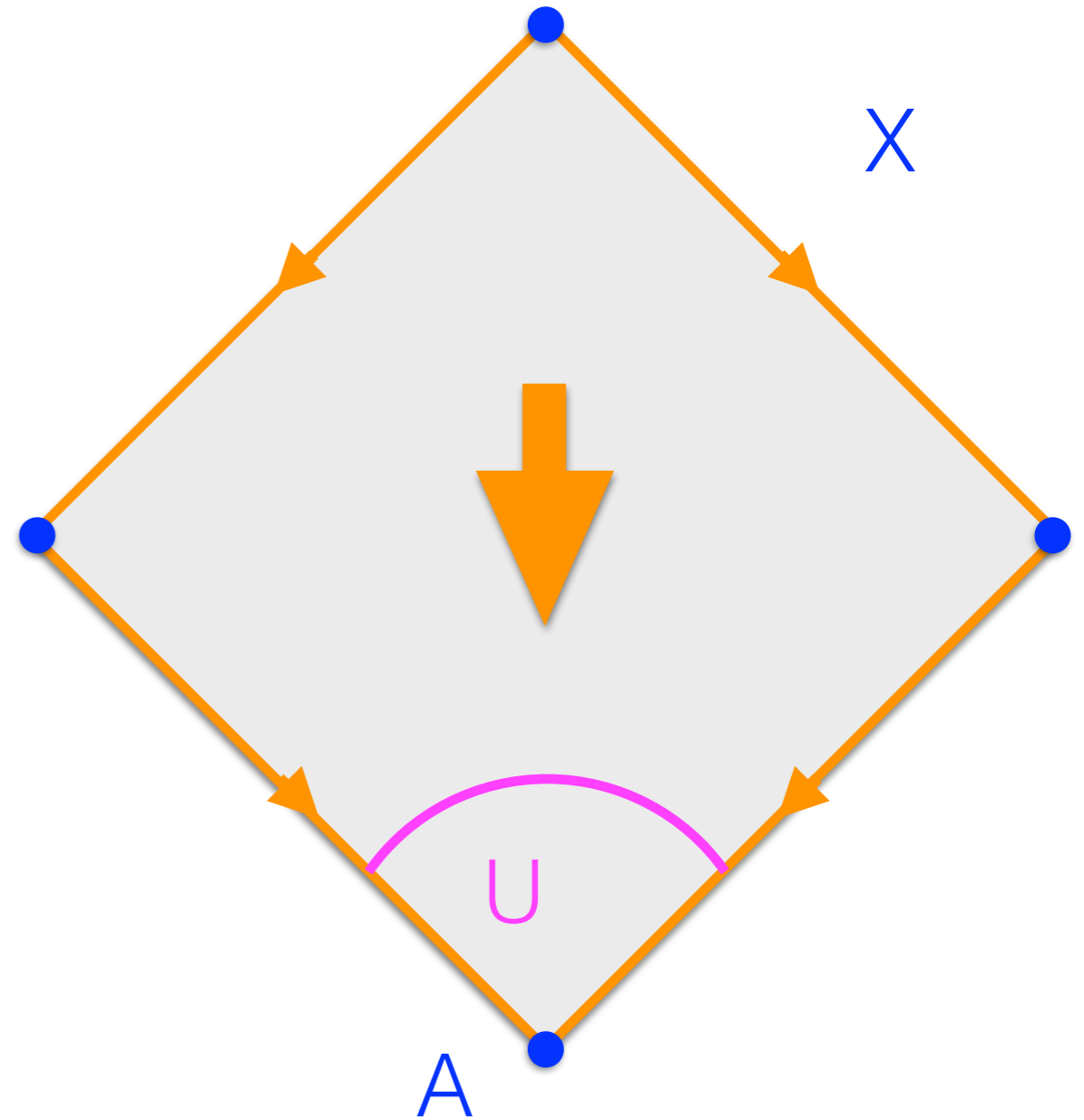
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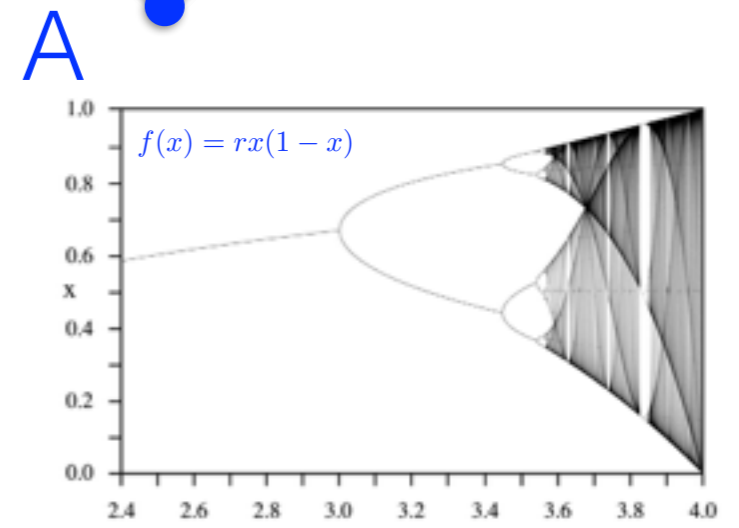
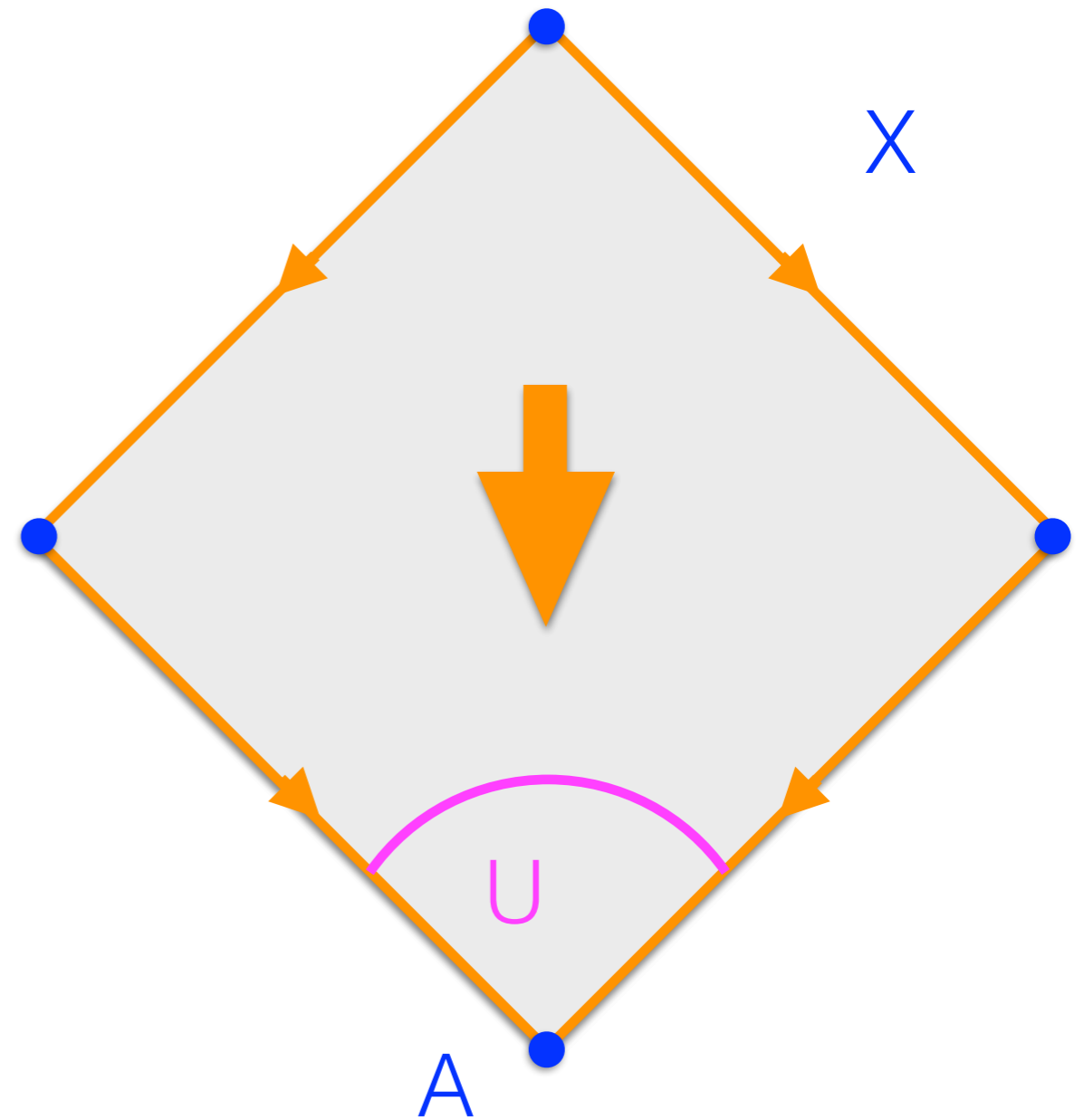
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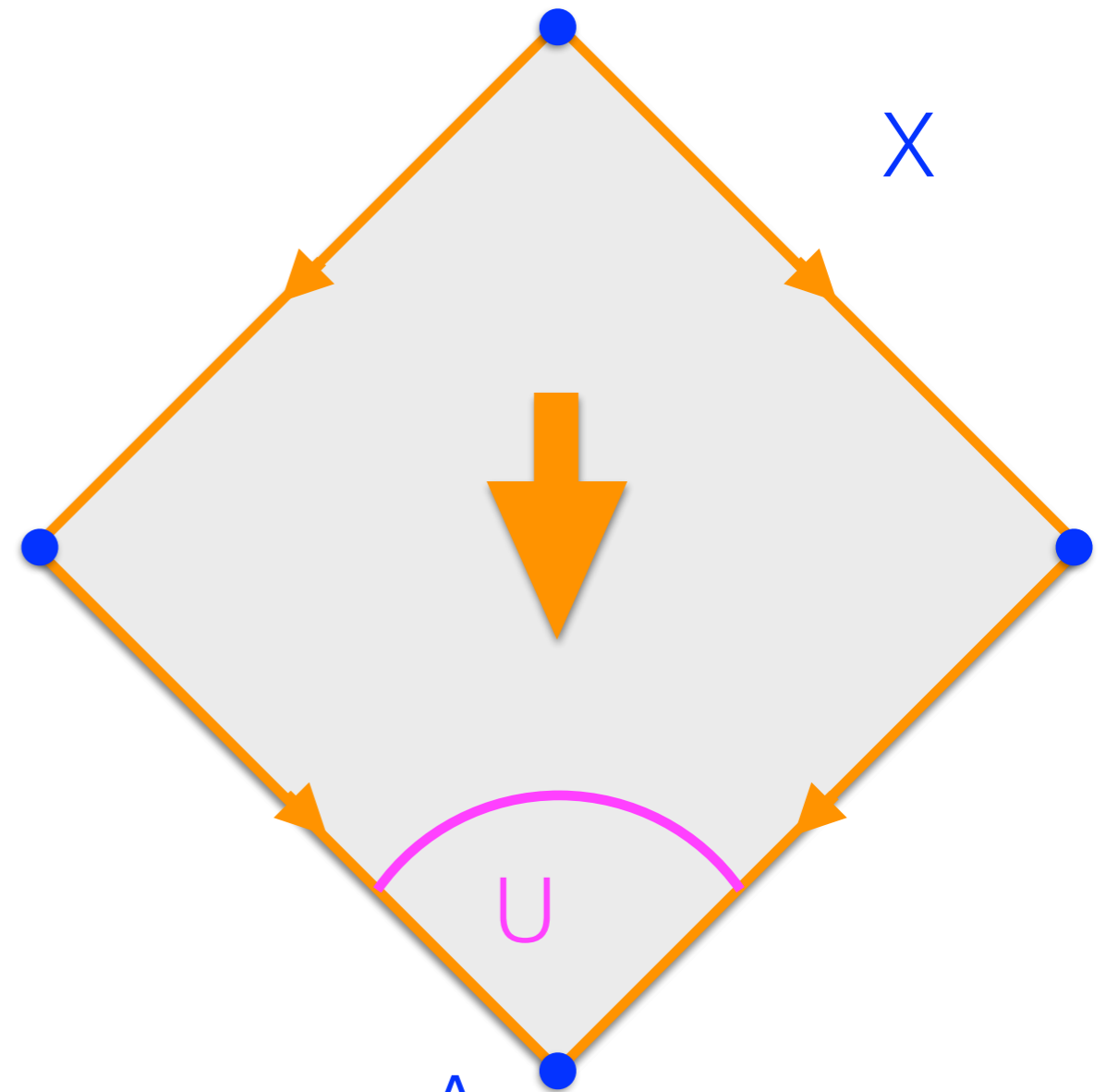
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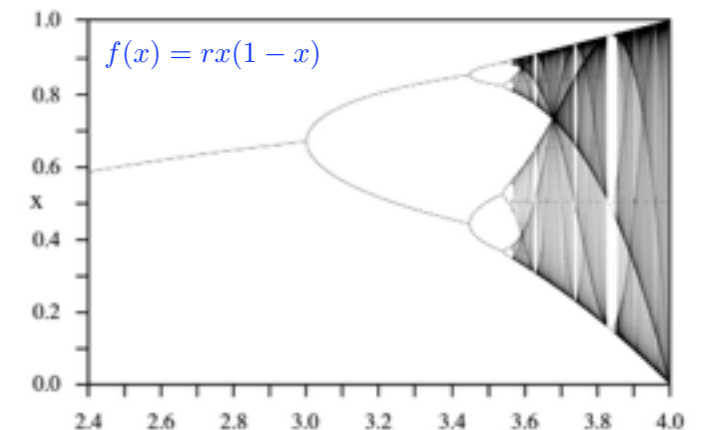
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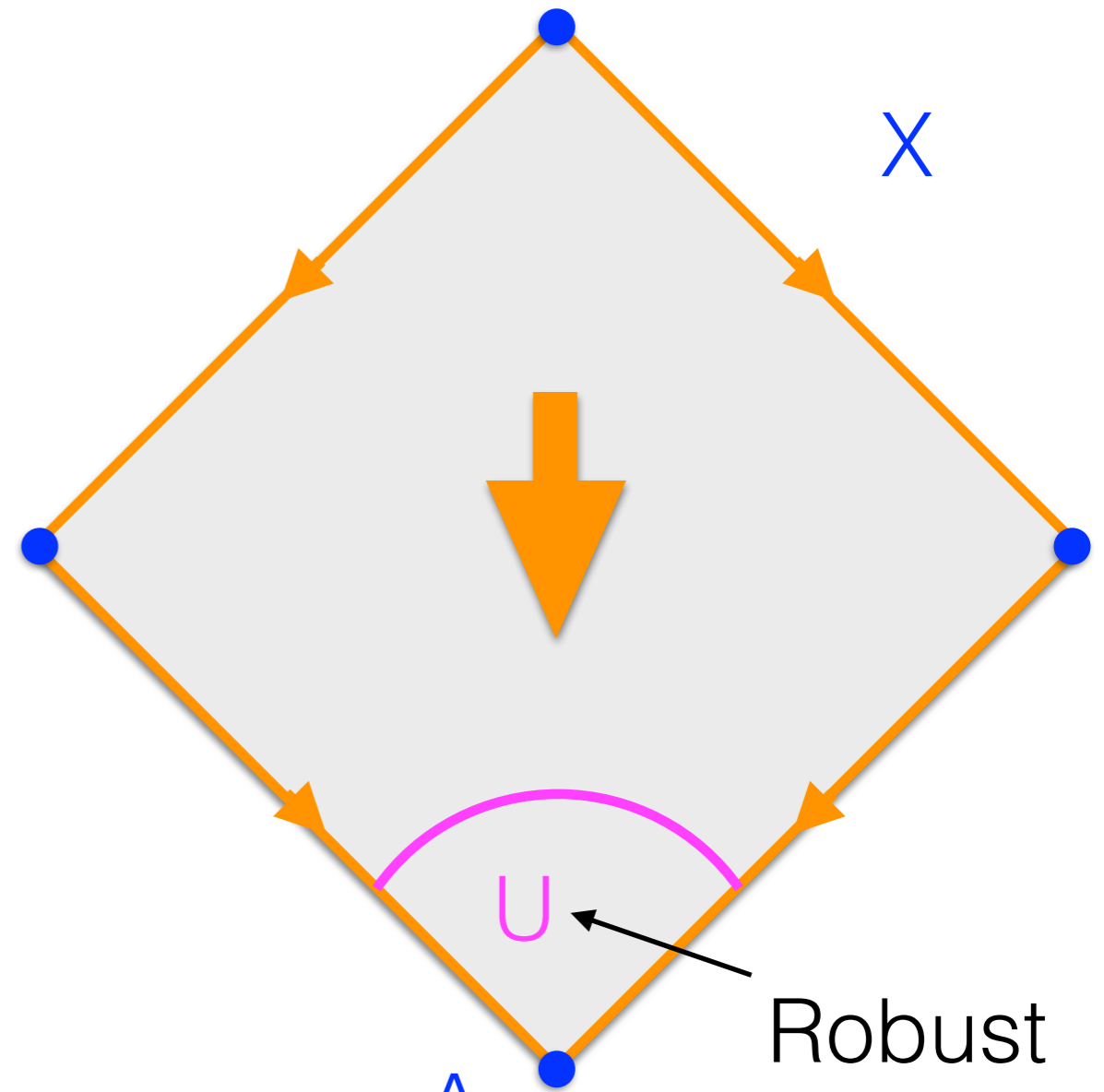
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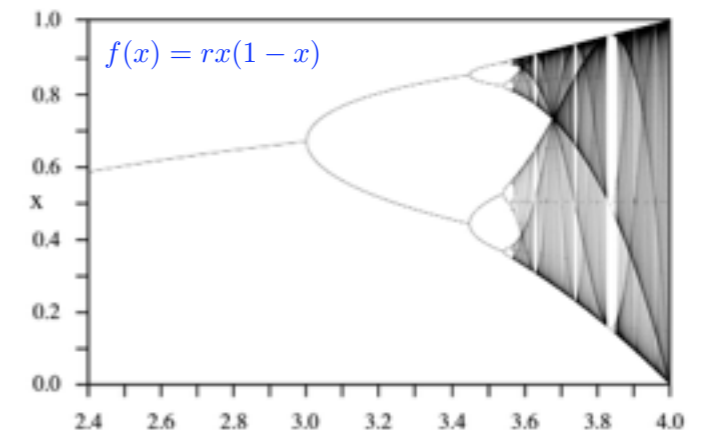
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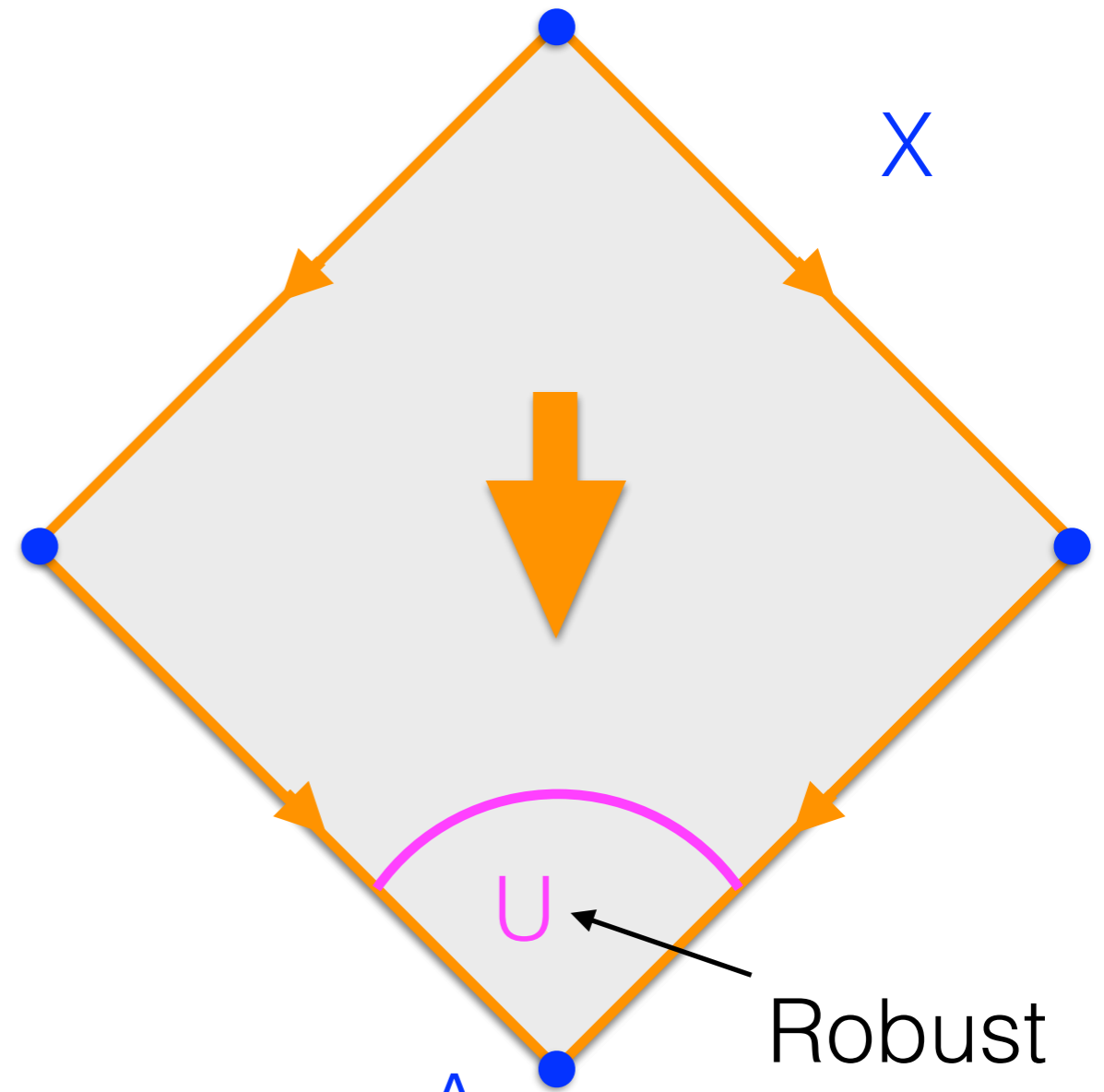


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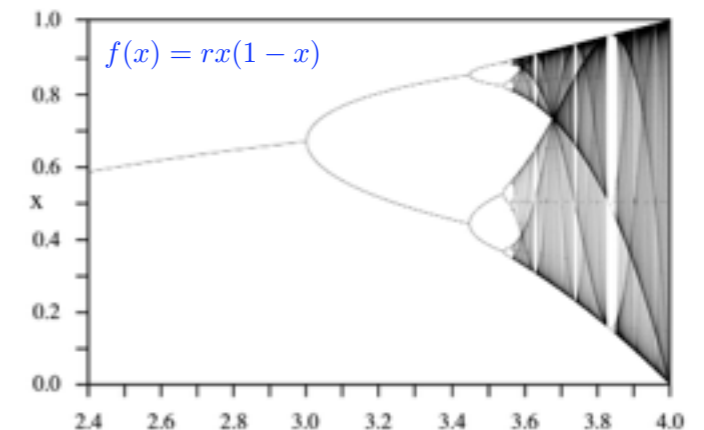
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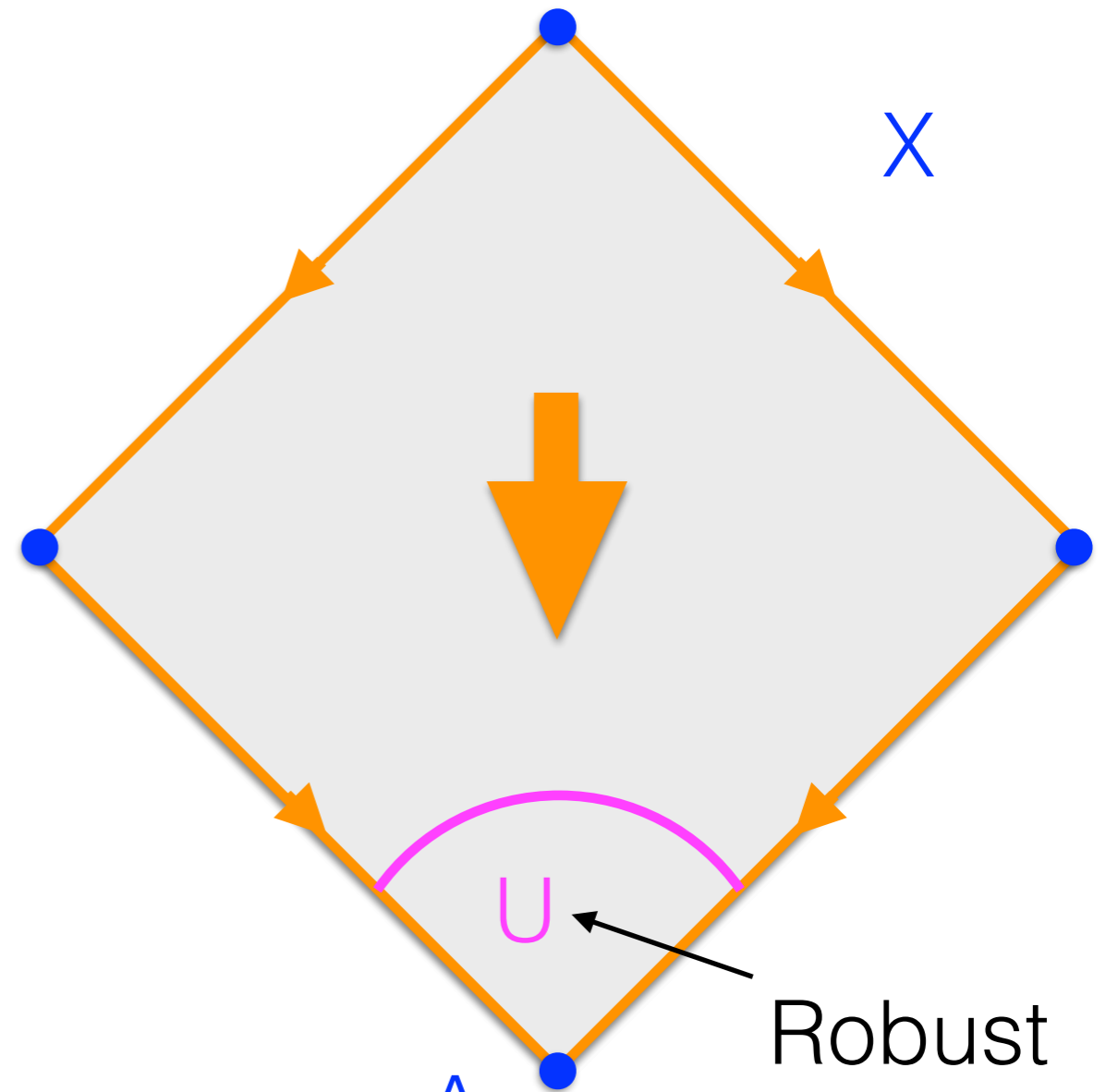
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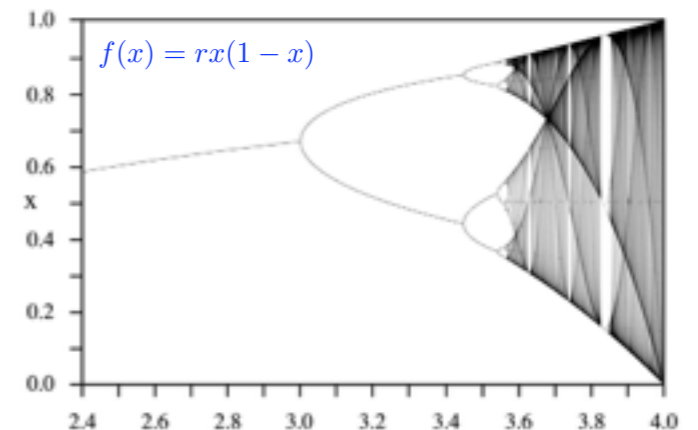
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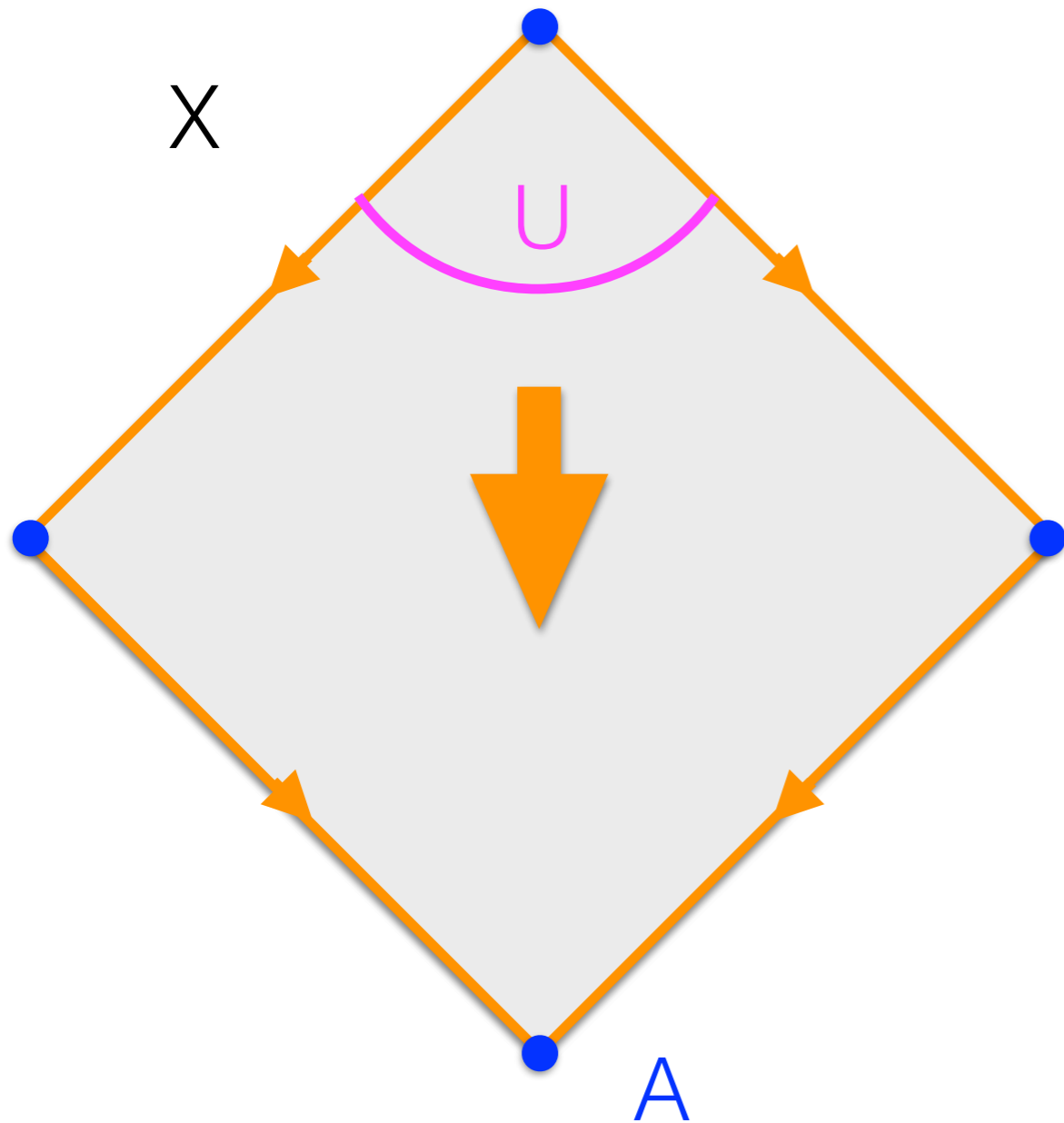
**Prop:**  $\text{Att}(f) \subset \text{Invset}(f)$ .



Not Robust  $\rightarrow$   $A$



# ALTERNATIVE CHARACTERIZATION OF CHAIN RECURRENT SET.

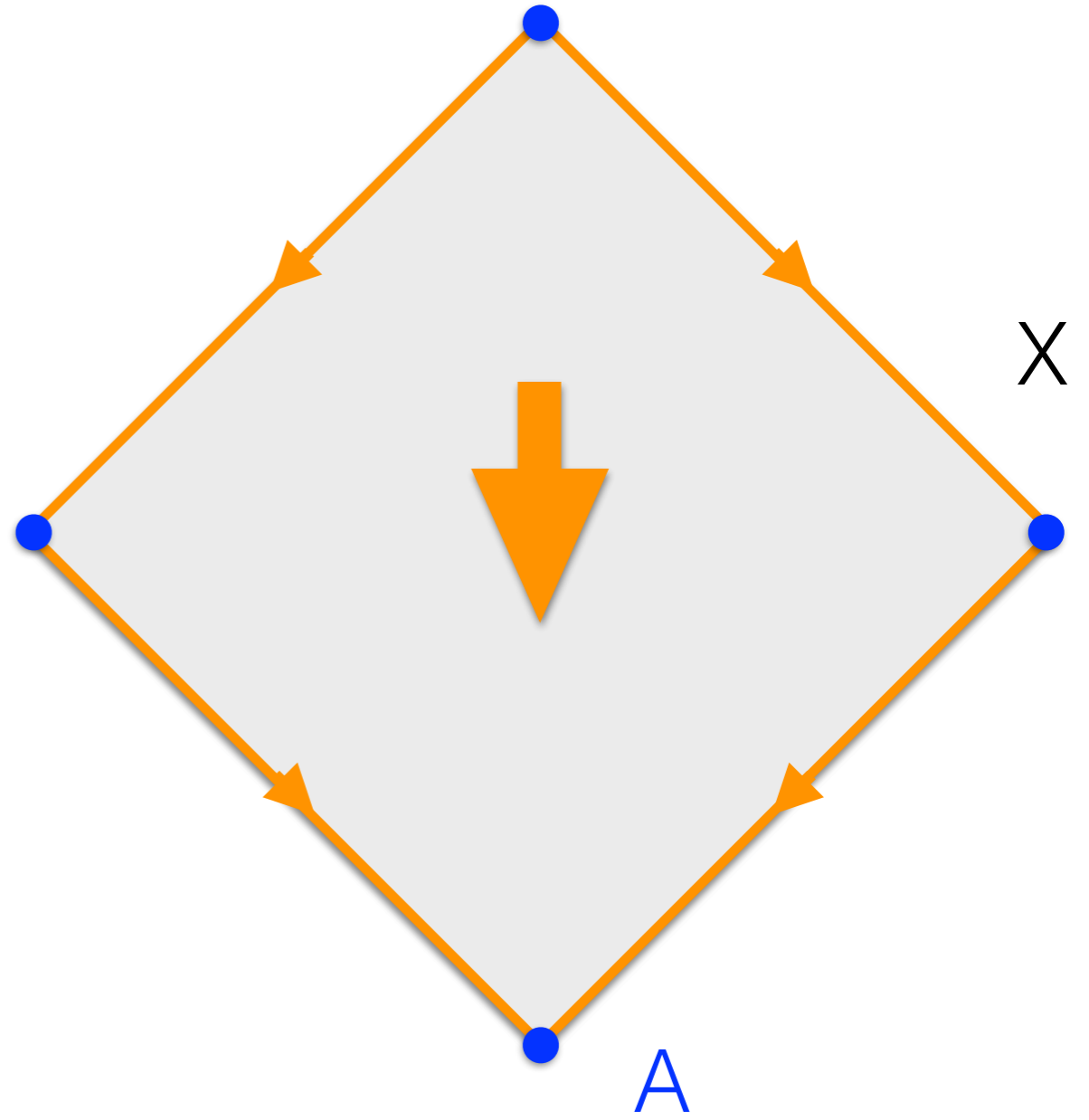


$R \subset X$  is an *repeller* if there exists a neighborhood  $U$  of  $R$  such that

$$R = \alpha(U, f) := \bigcap_{n \leq 0} \overline{\bigcup_{m \leq n} f^m(U)}$$

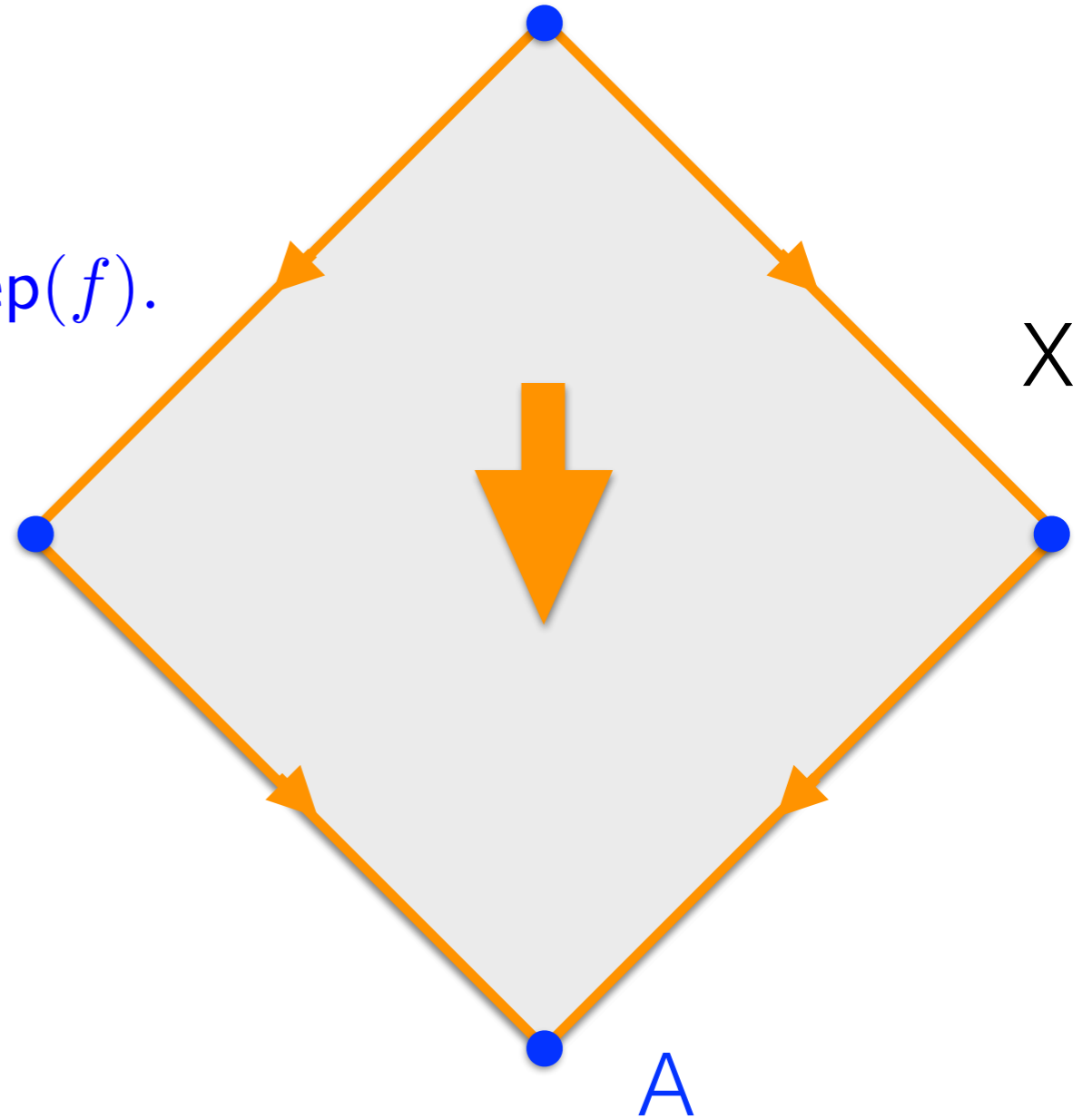
The set of repellers is denoted by  $\text{Rep}(f)$ .

**Prop:**  $\text{Rep}(f) \subset \text{Invset}^\pm(f)$ .



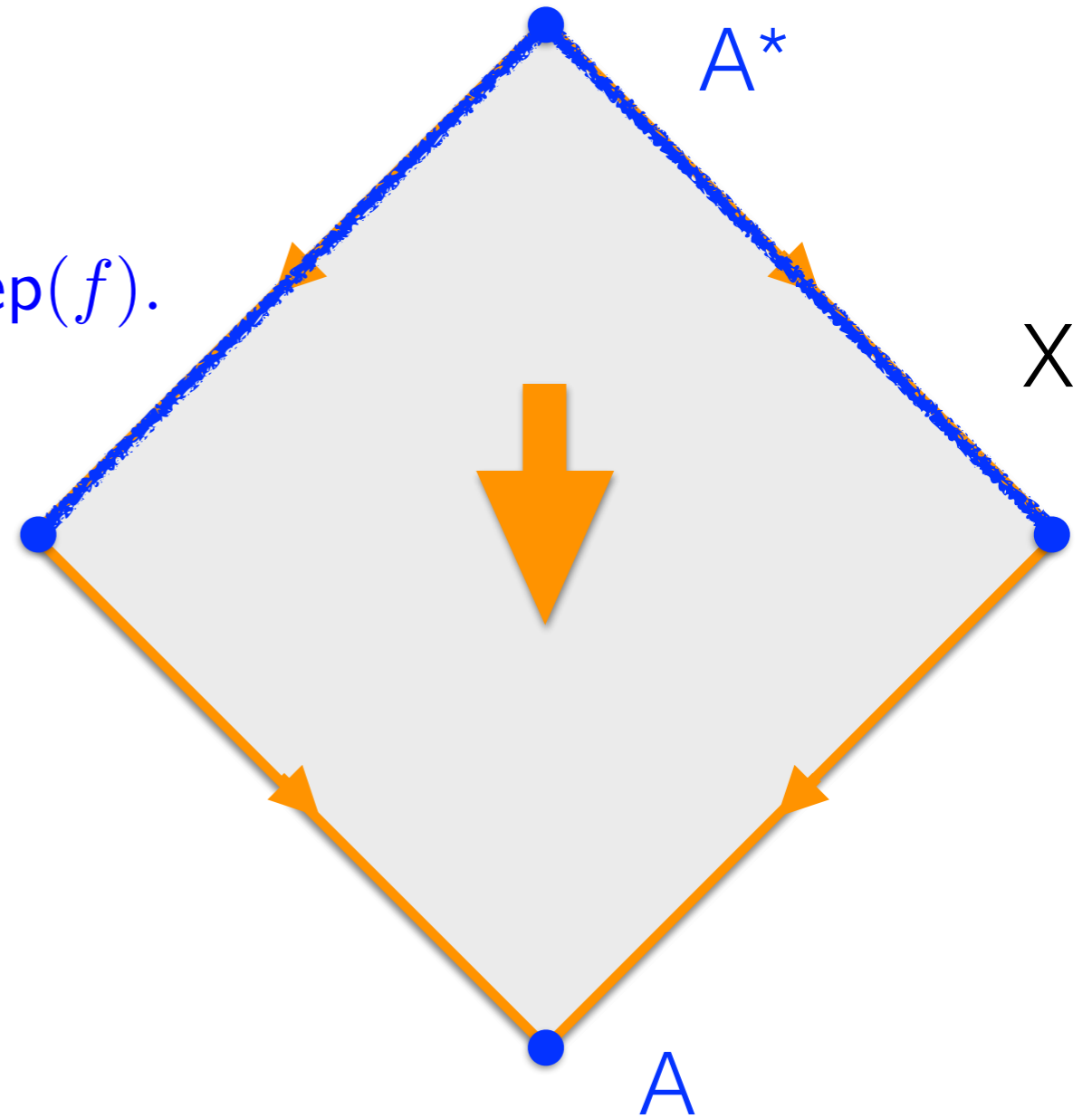
**Prop:** Given  $A \in \text{Att}(f)$  its *dual repeller* is

$$A^* := \{x \in X \mid \omega(x, f) \cap A = \emptyset\} \in \text{Rep}(f).$$



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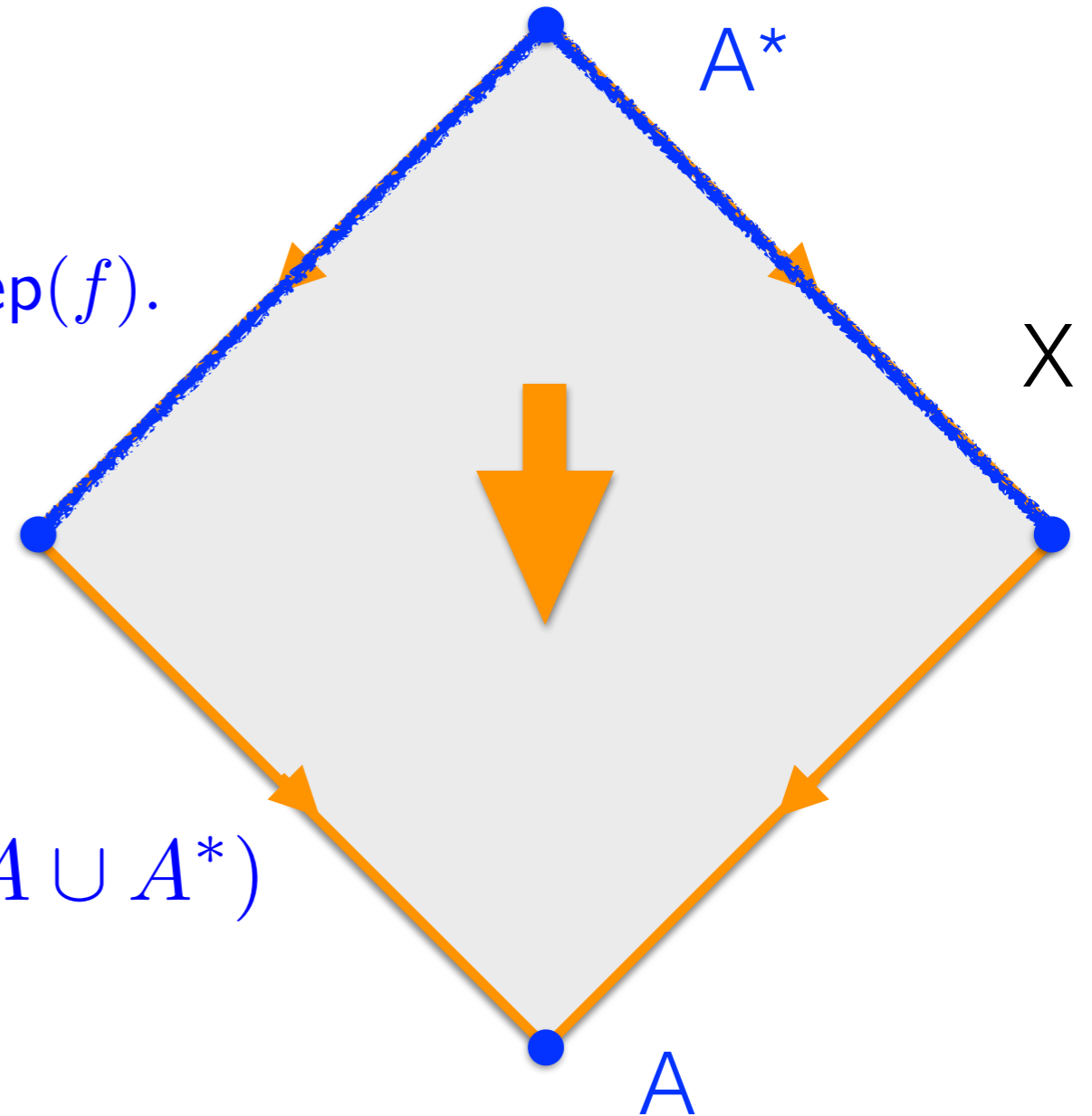


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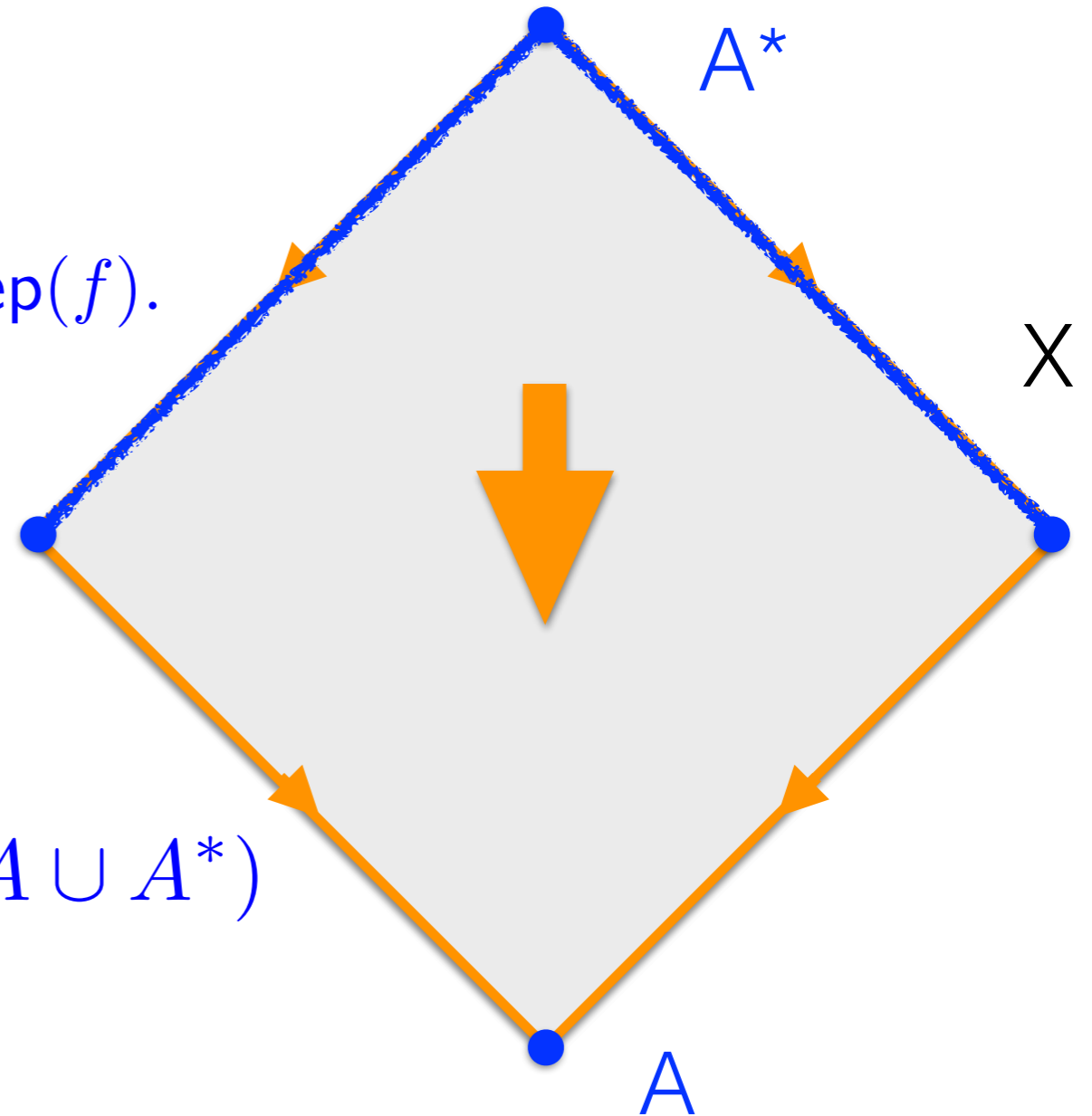
**Theorem:** (Conley)

$$\mathcal{R}(X, f) = \bigcap_{A \in \text{Att}(f)} (A \cup A^*)$$



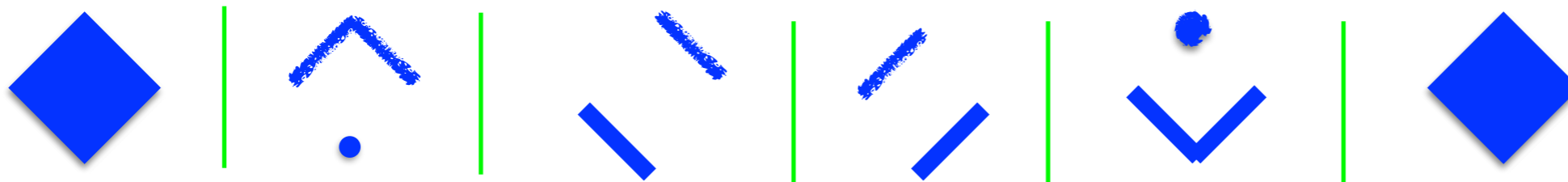
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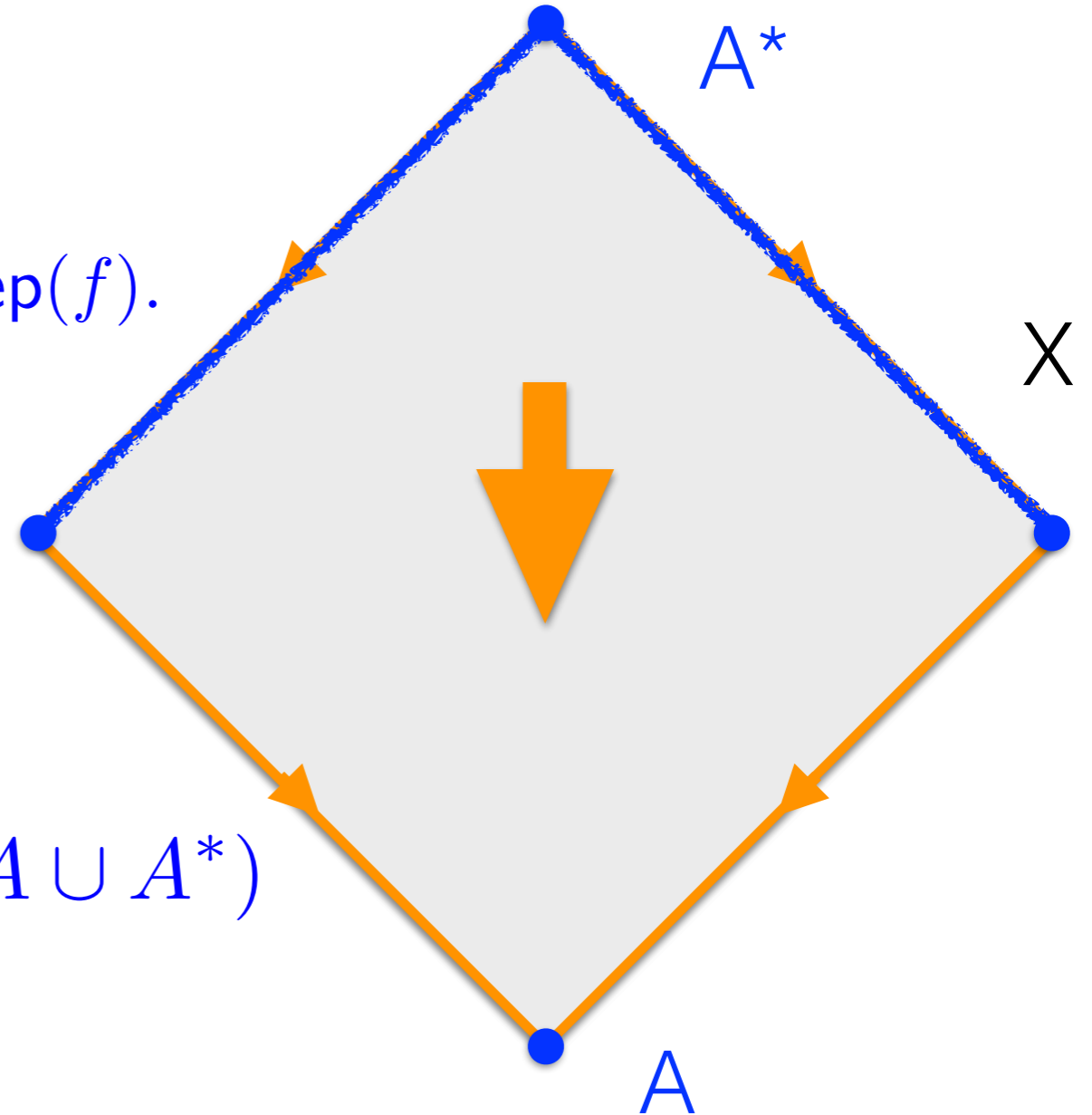
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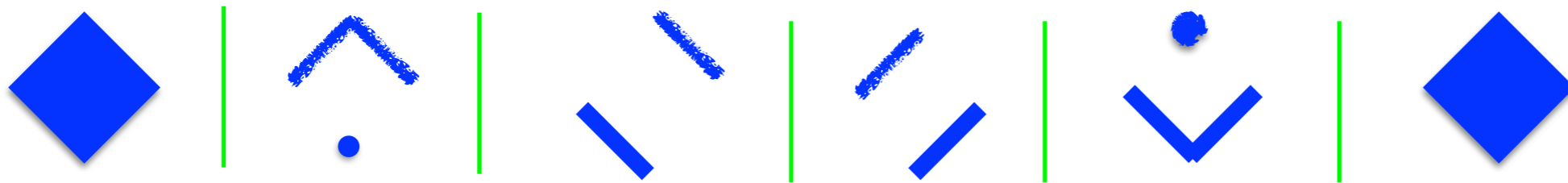
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Suggests the importance of understanding structure of  $\text{Att}(f)$  and  $\text{Rep}(f)$ .