

Spectral sequences  
downgraded to  
Elementary Gauss reductions

```
;; Clock  
Computing  
<TnPr <TnPr  
End of computing.  
  
;; Clock -> 2002-01-17, 19h 25m 36s.  
Computing the boundary of the generator 19 (dimension 7) :  
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>  
End of computing.  
  
Homology in dimension 6 :  
  
Component Z/12Z  
  
---done---  
  
;; Clock -> 2002-01-17, 19h 27m 15s
```

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MAP 2014, IHP Paris, May 2014*

## Semantics of colours:

Blue = “Standard” Mathematics

Red = Constructive, effective,

algorithm, machine object, ...

Violet = Problem, difficulty,

obstacle, disadvantage, ...

Green = Solution, essential point,

mathematicians, ...

## Plan.

1. Introduction.
2. Connection **HPT**  $\Rightarrow$  **SS**.
3. **Homological Reductions** and **HPT statement**.
4. **Hexagonal Lemma**.
5. **Algebraic HPT Proof**.
6. The **topological** case.

1/6. Introduction.
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SEMSS = Serre or Eilenberg-Moore Spectral Sequence

HPT = Homological Perturbation Theorem  
= Perturbation Lemma.

GR = Elementary Gauss Reduction  
(known by the Babylonians).

Theorem: GR  $\Rightarrow$  HPT  $\Rightarrow$  SEMSS.

2/6. Connection **HPT**  $\Rightarrow$  **SSS** = **Serre Spectral Seq.**

**SSS** = Info connecting  $H_*F$ ,  $H_*B$  and  $H_*E$

when  $E$  = total space of a fibration

of base space  $B$  and fibre space  $F$ .

$$E = F \times_{\tau} B \quad (\tau = \text{twisting function})$$

$$S^3 = S^1 \times_h S^2 \quad (h = \text{Hopf twisting function})$$

Proof of **HPT**  $\Rightarrow$  **SSS**:

1. Start with  $\tau_0$  = trivial twisting = no twisting at all

$$\text{Eilenberg-Zilber Theorem} \Rightarrow [H_*B + H_*F \Rightarrow H_*E_0]$$

2. **HPT** = **Implicit Function** Theorem  $\Rightarrow$

$$[\tau \text{ close to } \tau_0] \Rightarrow [H_*E_0 + \text{HPT} \Rightarrow H_*E].$$

Remark: **SSS** is not an algorithm

$$H_*B + H_*F \Rightarrow H_*(F \times_\tau B).$$

**HPT** is an algorithm  $H_*B + H_*F \Rightarrow H_*(F \times_\tau H_*B)$

$$\text{Ana Romero's thesis} \Rightarrow [\text{HPT} \Rightarrow \text{SSS}].$$

Theorem: **GR** +  $[(1 - x) \text{ invertible if } |x| < 1]$   $\Rightarrow$  **HPT**.

### 3/6. Homological Reductions and HPT Statement.

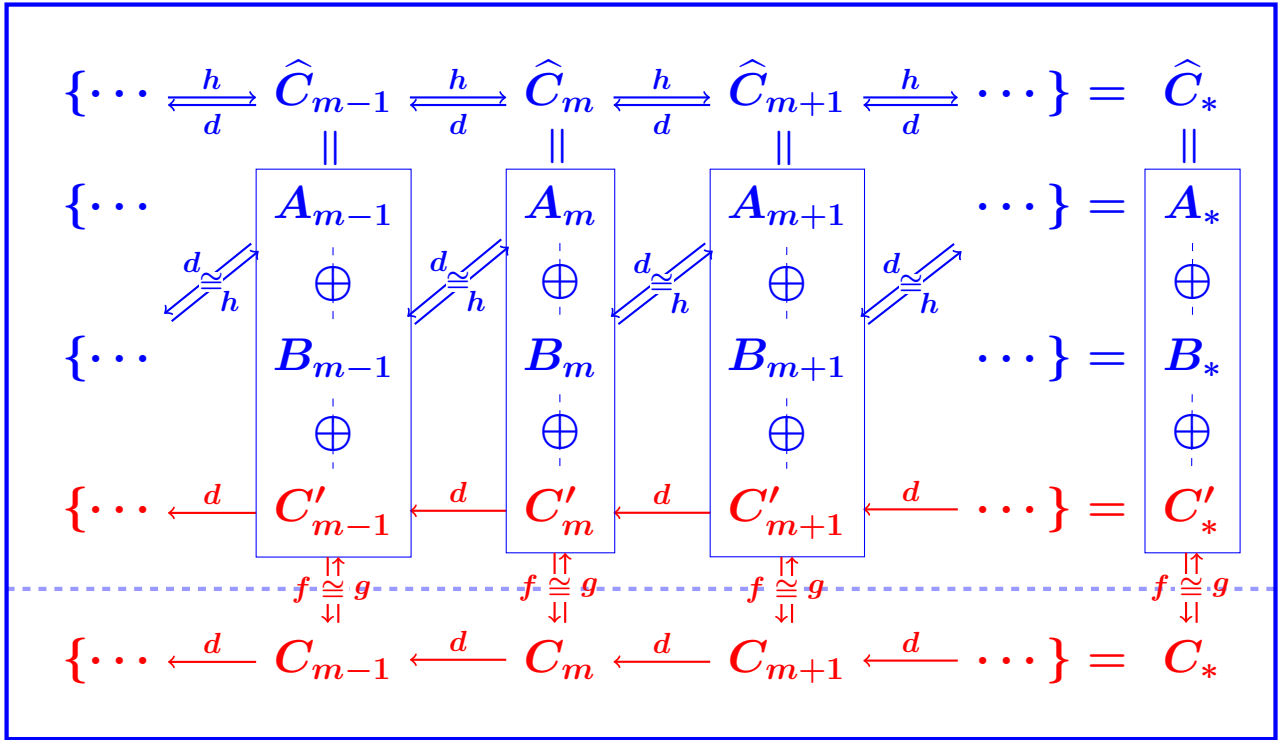
Definition: A (homological) reduction is a diagram:

$$\rho = (f, g, h) = \boxed{h \circlearrowleft (\widehat{C}_*, \widehat{d}_*) \begin{array}{c} \xleftarrow{g} \\ \xrightarrow{f} \end{array} (C_*, d_*)}$$

with:

1.  $\widehat{C}_*$  and  $C_*$  = chain complexes.
2.  $f$  and  $g$  = chain complex morphisms.
3.  $h$  = homotopy operator (degree +1).
4.  $fg = \text{id}_{C_*}$  and  $d_{\widehat{C}}h + hd_{\widehat{C}} + gf = \text{id}_{\widehat{C}_*}$ .
5.  $fh = 0$ ,  $hg = 0$  and  $hh = 0$ .

# Meaning = Reduction Diagram:





## Homological Perturbation Theorem (HPT)

Definition:  $(C_*, d)$  = given chain complex.

A **perturbation**  $\delta : C_* \rightarrow C_{*-1}$  is an operator of degree -1

satisfying  $(d + \delta)^2 = 0$  ( $\Leftrightarrow (d\delta + \delta d + \delta^2) = 0$ ):

$$(C_*, d) + (\delta) \mapsto (C_*, d + \delta).$$

Let  $\rho : h \hookrightarrow (\widehat{C}_*, \widehat{d}_*) \xrightleftharpoons[f]{g} (C_*, d_*)$  be a given reduction

and  $\widehat{\delta}$  a **perturbation** of  $\widehat{d}$

satisfying  $h\widehat{\delta}$  pointwise nilpotent.

Theorem: The **HPT** determines a **new reduction**:

$$\rho' : h + \delta_h \hookrightarrow (\widehat{C}_*, \widehat{d}_* + \widehat{\delta}_*) \xrightleftharpoons[f + \delta_f]{g + \delta_g} (C_*, d_* + \delta_{d_*})$$

4/6. Hexagonal Lemma.

$R =$  Unitary ring

$\varepsilon, \varphi, \psi, \beta \in R$  with  $\varepsilon$  invertible.

Gauss discussion of (1) + (2):

$$(1) \quad \varepsilon x + \varphi y = a$$

$$(2) \quad \psi x + \beta y = b$$

$$(2) - \psi\varepsilon^{-1}(1) \Rightarrow$$

$$(2') \quad (\beta - \psi\varepsilon^{-1}\varphi)y = (b - \psi\varepsilon^{-1}a)$$

$\Rightarrow$  (1) + (2) has a solution  $\Leftrightarrow$

$$(\beta - \psi\varepsilon^{-1}\varphi) \mid (b - \psi\varepsilon^{-1}a) \Rightarrow y = \dots$$

$$\Rightarrow x = \varepsilon^{-1}a - \varepsilon^{-1}\varphi y$$

Matrix translation:

$$\begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$\Leftrightarrow$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ -\psi\varepsilon^{-1} & 1 \end{pmatrix} \begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \begin{pmatrix} 1 & -\varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} \varepsilon & 0 \\ 0 & \beta - \psi\varepsilon^{-1}\varphi \end{pmatrix}} \underbrace{\begin{pmatrix} 1 & \varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}_{\begin{pmatrix} x + \varepsilon^{-1}\varphi y \\ y \end{pmatrix}} = \underbrace{\begin{pmatrix} 1 & 0 \\ -\psi\varepsilon^{-1} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}}_{\begin{pmatrix} a \\ -\psi\varepsilon^{-1}a + b \end{pmatrix}}$$

$\Leftrightarrow$

$$\varepsilon(x + \varepsilon^{-1}\varphi y) = a$$

$$(\beta - \psi\varepsilon^{-1}\varphi)y = (b - \psi\varepsilon^{-1}a)$$

$\Leftrightarrow$

$$(\beta - \psi\varepsilon^{-1}\varphi) \mid (b - \psi\varepsilon^{-1}a) \Rightarrow \dots$$

Diagram translation:

$$\begin{array}{ccc}
 & \begin{pmatrix} 1 & -\varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix} & \\
 & \begin{array}{c} \leftarrow \\ \rightarrow \end{array} & \\
 R^2 & & R^2 \\
 & \begin{pmatrix} 1 & \varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix} & \\
 \begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \downarrow & & \downarrow \begin{pmatrix} \varepsilon & 0 \\ 0 & \beta - \psi\varepsilon^{-1}\varphi \end{pmatrix} \\
 R^2 & \begin{pmatrix} 1 & -0 \\ \psi\varepsilon^{-1} & 1 \end{pmatrix} & R^2 \\
 & \begin{array}{c} \leftarrow \\ \rightarrow \end{array} & \\
 & \begin{pmatrix} 1 & -0 \\ -\psi\varepsilon^{-1} & 1 \end{pmatrix} &
 \end{array}$$

Combined with an obvious reduction:

$$\begin{array}{ccccc}
 & & \begin{pmatrix} 1 & -\varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix} & & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 R^2 & \xleftrightarrow{\hspace{10em}} & R^2 & \xleftrightarrow{\hspace{10em}} & R \\
 & & \begin{pmatrix} 1 & \varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix} & & \begin{pmatrix} 0 & 1 \end{pmatrix} \\
 \begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} & & h = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 0 \end{pmatrix} & & \begin{pmatrix} \varepsilon & 0 \\ 0 & \beta - \psi\varepsilon^{-1}\varphi \end{pmatrix} & & (\beta - \psi\varepsilon^{-1}\varphi) \\
 & & \begin{pmatrix} 1 & 0 \\ \psi\varepsilon^{-1} & 1 \end{pmatrix} & & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 R^2 & \xleftrightarrow{\hspace{10em}} & R^2 & \xleftrightarrow{\hspace{10em}} & R \\
 & & \begin{pmatrix} 1 & 0 \\ -\psi\varepsilon^{-1} & 1 \end{pmatrix} & & \begin{pmatrix} 0 & 1 \end{pmatrix}
 \end{array}$$

⇒

$\Rightarrow$  Canonical **reduction** induced by  $\varepsilon$  invertible

$$\begin{array}{ccc}
 & & g = \begin{pmatrix} -\varepsilon^{-1}\varphi \\ 1 \end{pmatrix} \\
 & \xleftarrow{\hspace{10em}} & \xrightarrow{\hspace{10em}} \\
 R^2 & & R \\
 & \xleftarrow{\hspace{10em}} & \xrightarrow{\hspace{10em}} \\
 & & f = \begin{pmatrix} 0 & 1 \end{pmatrix} \\
 & & \downarrow \\
 & & (\beta - \psi\varepsilon^{-1}\varphi) \\
 & & \downarrow \\
 & & R \\
 & & \downarrow \\
 & & R \\
 & & \downarrow \\
 & & g = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 & \xleftarrow{\hspace{10em}} & \xrightarrow{\hspace{10em}} \\
 R^2 & & R \\
 & \xleftarrow{\hspace{10em}} & \xrightarrow{\hspace{10em}} \\
 & & f = \begin{pmatrix} -\psi\varepsilon^{-1} & 1 \end{pmatrix}
 \end{array}$$

$$h = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix}$$

The same is valid with

$$R^2 = R \oplus R \text{ replaced by } A_n \oplus B_n = C_n$$

$$\text{or by } A_{n-1} \oplus B_{n-1} = C_{n-1}$$

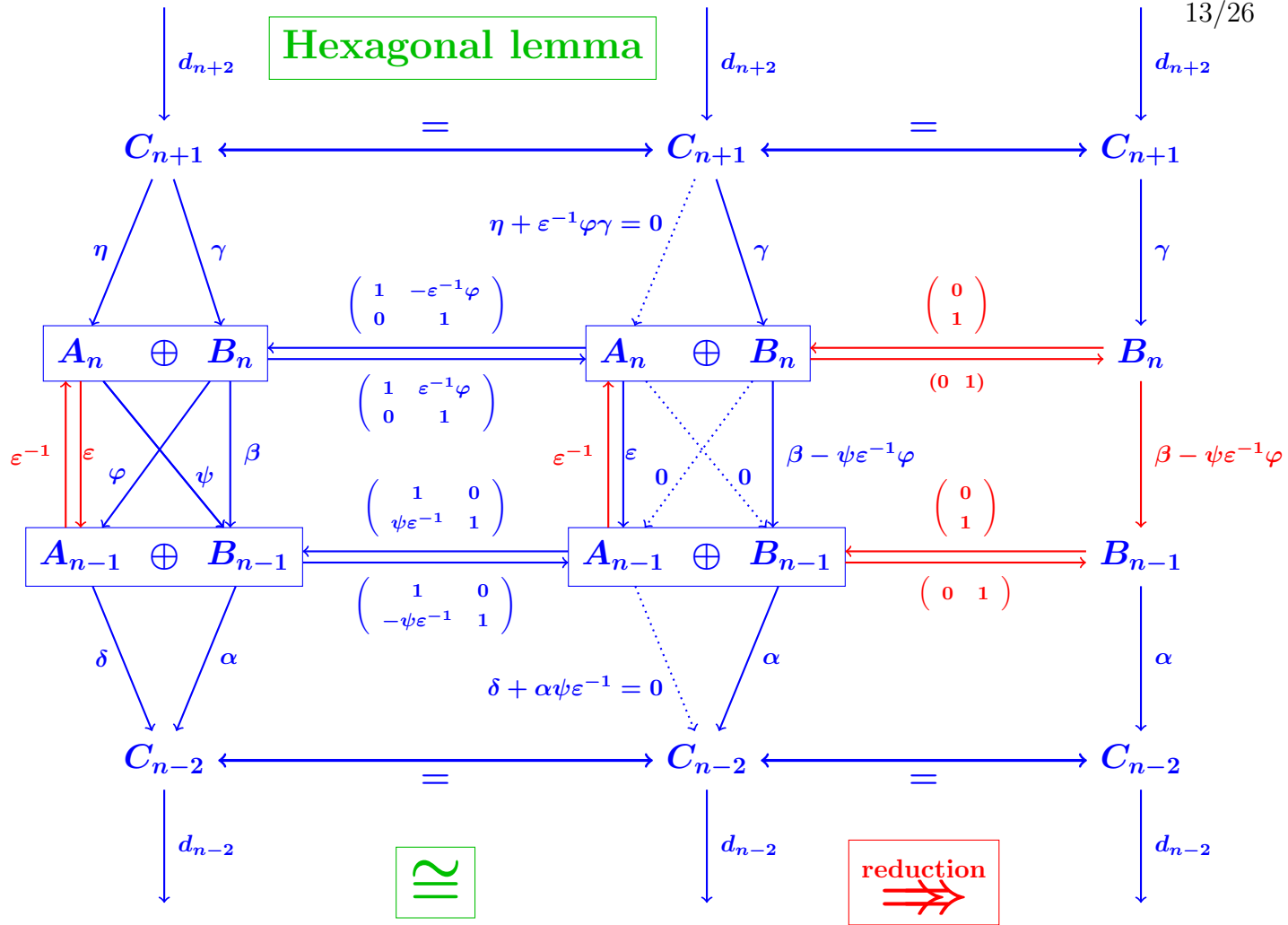
and:

$$\begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} : A_n \oplus B_n \rightarrow A_{n-1} \oplus B_{n-1}$$

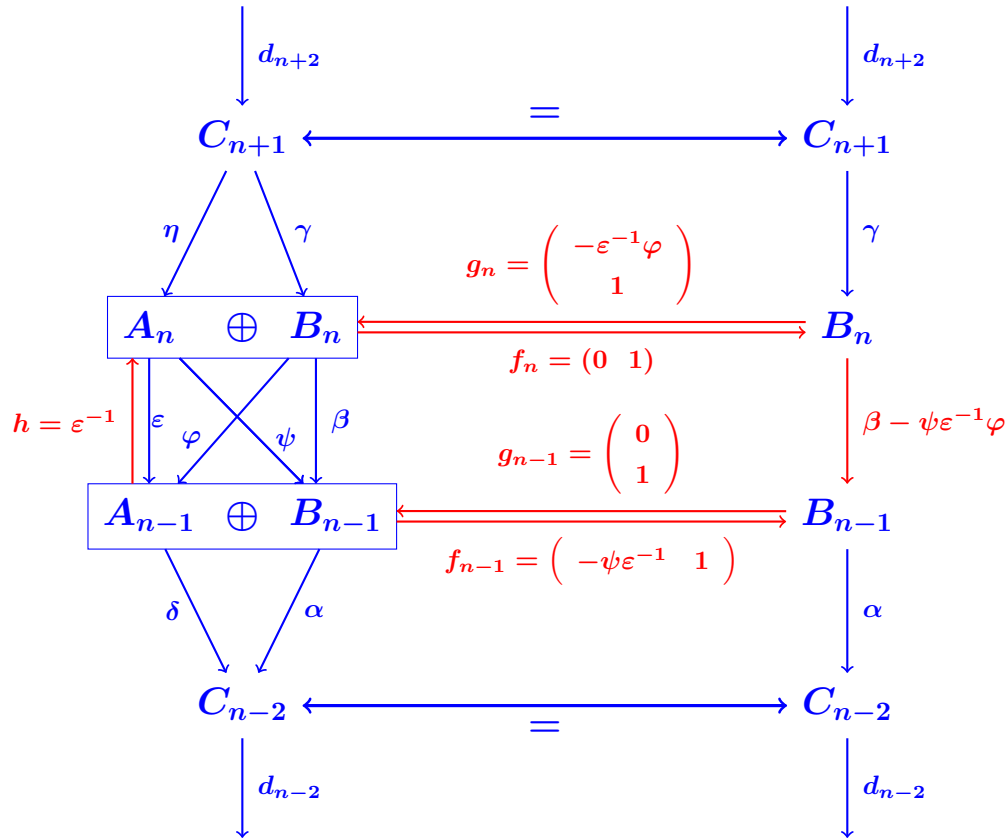
with  $\varepsilon : A_n \rightarrow A_{n-1}$  isomorphism.

$\Rightarrow$  Hexagonal lemma.

Hexagonal lemma

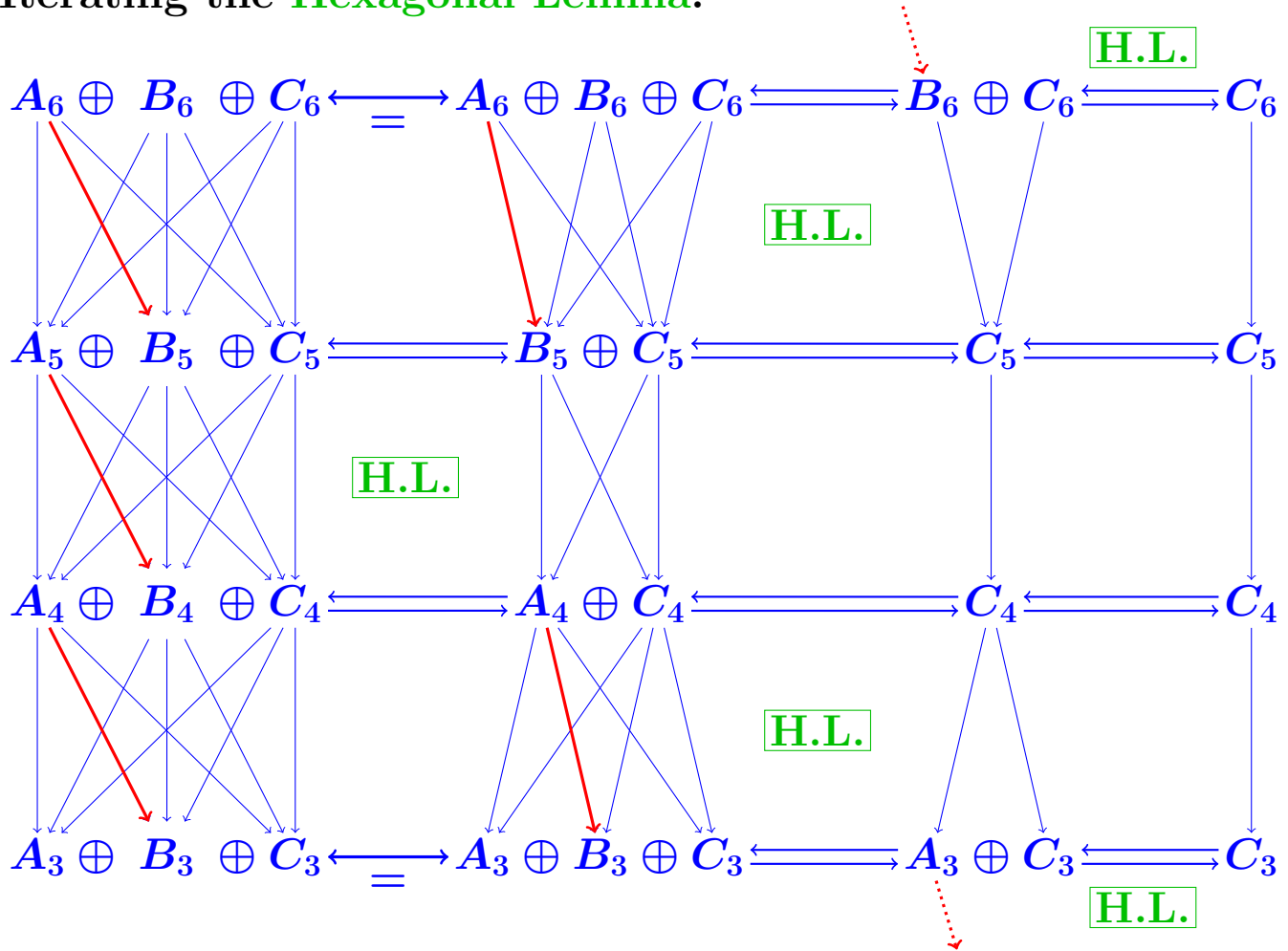






Hexagonal lemma

# Iterating the Hexagonal Lemma:



Final result:

$$\begin{array}{c}
 \begin{array}{ccc}
 A_5 & \oplus & B_5 & \oplus & C_5 \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 A_4 & \oplus & B_4 & \oplus & C_4
 \end{array}
 \end{array}$$

$d^5$  (vertical arrow from  $A_5$  to  $A_4$ )  
 $d_{21}^5$  (red diagonal arrow from  $A_5$  to  $B_4$ )

$$d^5 = \begin{pmatrix} d_{11}^5 & d_{12}^5 & d_{13}^5 \\ d_{21}^5 & d_{22}^5 & d_{23}^5 \\ d_{31}^5 & d_{32}^5 & d_{33}^5 \end{pmatrix}$$

$$g_5 = \begin{pmatrix} -d_{21}^5 & -1 & d_{23}^5 \\ 0 & & \\ 1 & & \end{pmatrix}$$

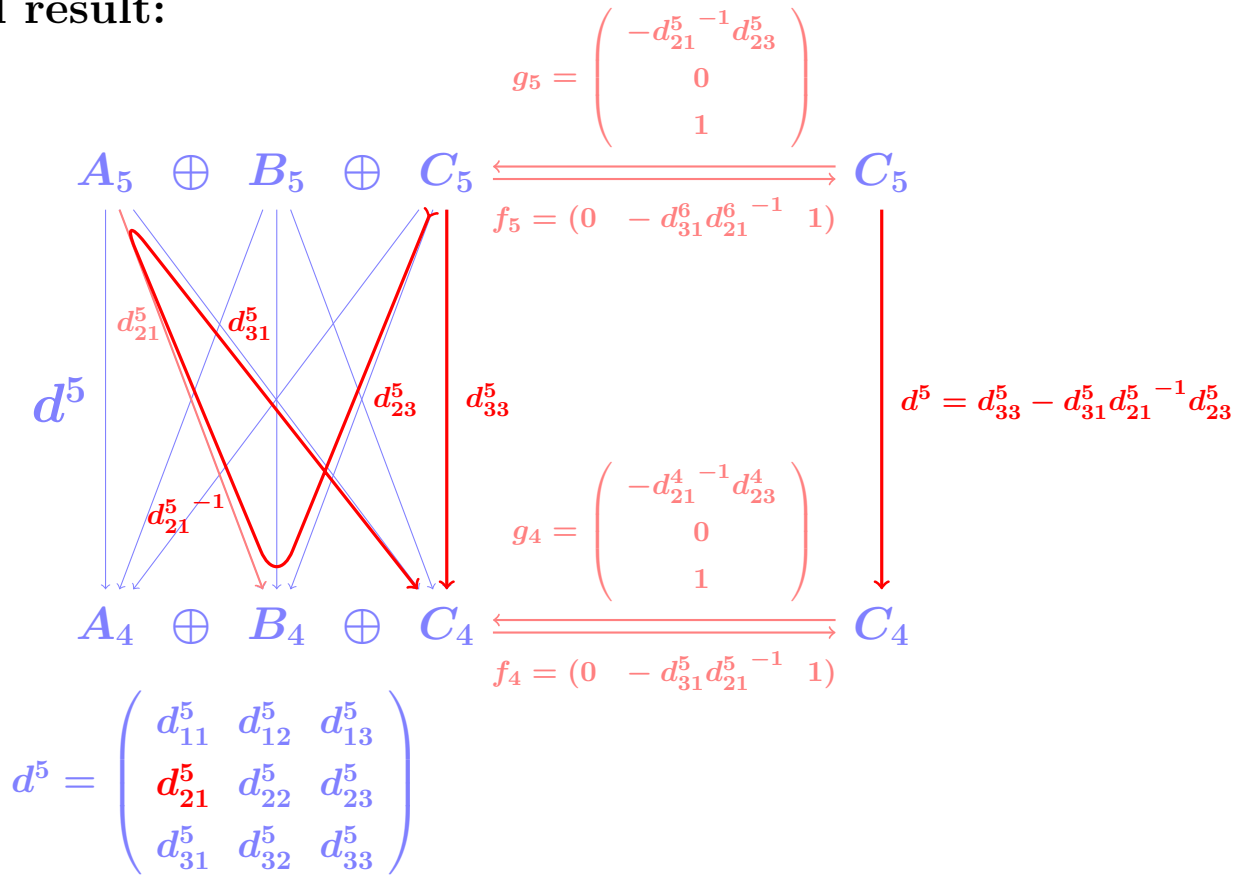
$$f_5 = (0 \quad -d_{31}^6 d_{21}^6 & -1 \quad 1)$$

$$g_4 = \begin{pmatrix} -d_{21}^4 & -1 & d_{23}^4 \\ 0 & & \\ 1 & & \end{pmatrix}$$

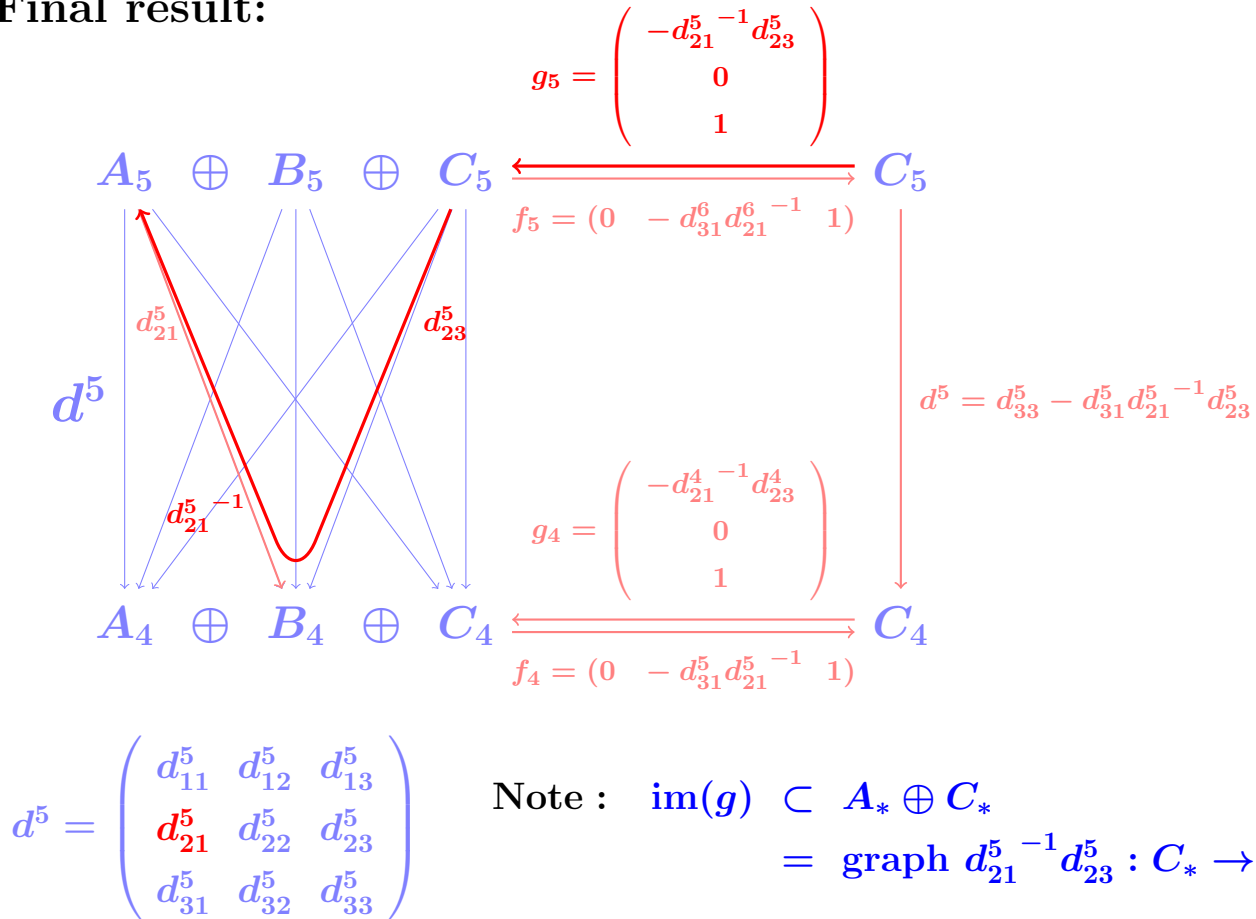
$$f_4 = (0 \quad -d_{31}^5 d_{21}^5 & -1 \quad 1)$$

$$d^5 = d_{33}^5 - d_{31}^5 d_{21}^5 & -1 \quad d_{23}^5$$

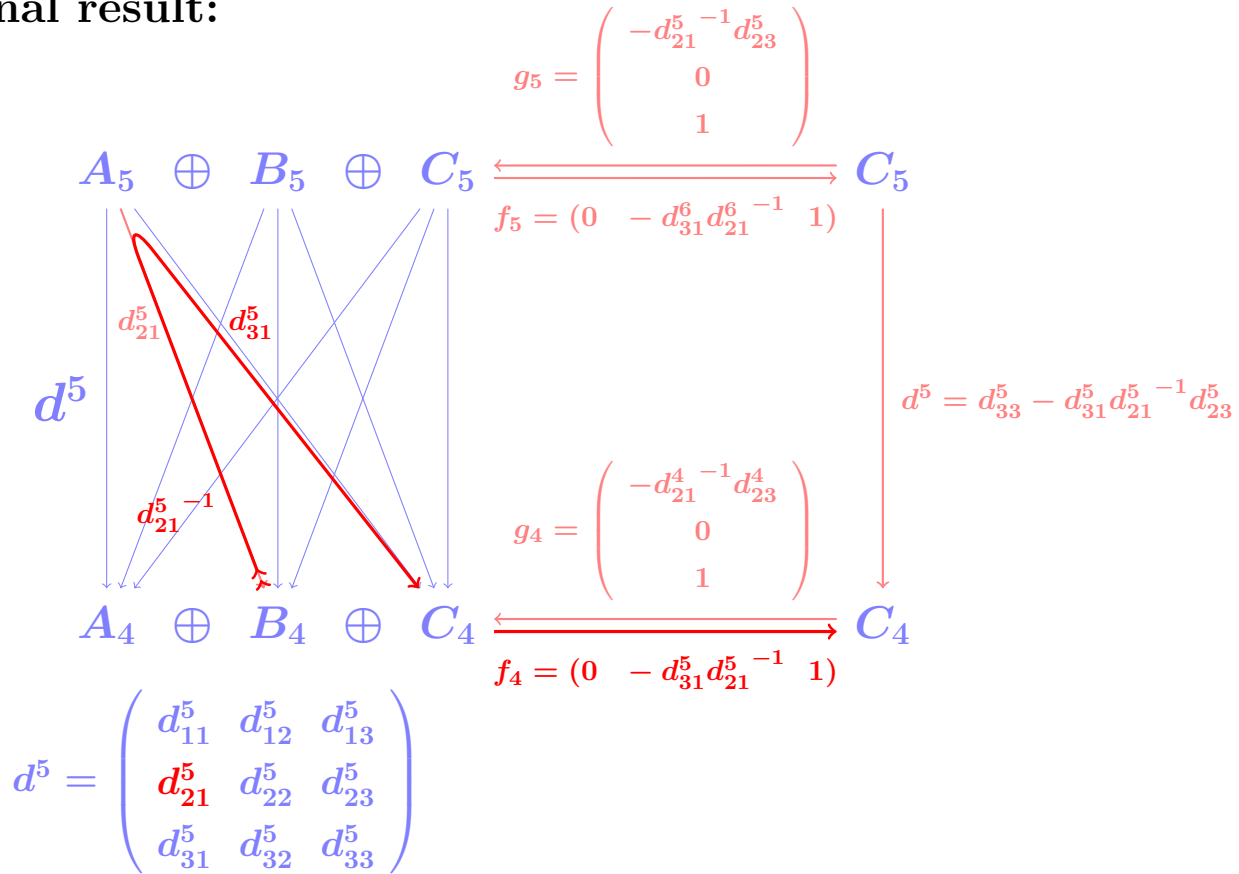
Final result:



Final result:



Final result:



## Global Hexagonal Theorem:

Input: A chain complex  $(C_*, d_*)$

with for every  $n \in \mathbb{Z}$  a decomposition:

$$C_n = C_n^1 \oplus C_n^2 \oplus C_n^3 \quad d_n = \begin{pmatrix} d_{n,11} & d_{n,12} & d_{n,13} \\ \mathbf{d}_{n,21} & d_{n,22} & d_{n,23} \\ d_{n,31} & d_{n,32} & d_{n,33} \end{pmatrix}$$

with  $\mathbf{d}_{n,21} : C_n^1 \rightarrow C_{n-1}^2$  isomorphism  $\forall n$ .

Output: A canonical reduction:

$$(C_*, d_*) = (C_*^1 \oplus C_*^2 \oplus C_*^3, d_*) \Rightarrow (C_*^3, d'_*)$$

5/6. Algebraic HPT Proof.

Definition:  $(C_*, d)$  = given chain complex.

A perturbation  $\delta : C_* \rightarrow C_{*-1}$  is an operator of degree -1

satisfying  $(d + \delta)^2 = 0$  ( $\Leftrightarrow (d\delta + \delta d + \delta^2) = 0$ ):

$$(C_*, d) + (\delta) \mapsto (C_*, d + \delta).$$

Let  $\rho : h \circlearrowleft (\widehat{C}_*, \widehat{d}_*) \xrightleftharpoons[f]{g} (C_*, d_*)$  be a given reduction

and  $\widehat{\delta}$  a perturbation of  $\widehat{d}$

satisfying  $h\widehat{\delta}$  pointwise nilpotent.

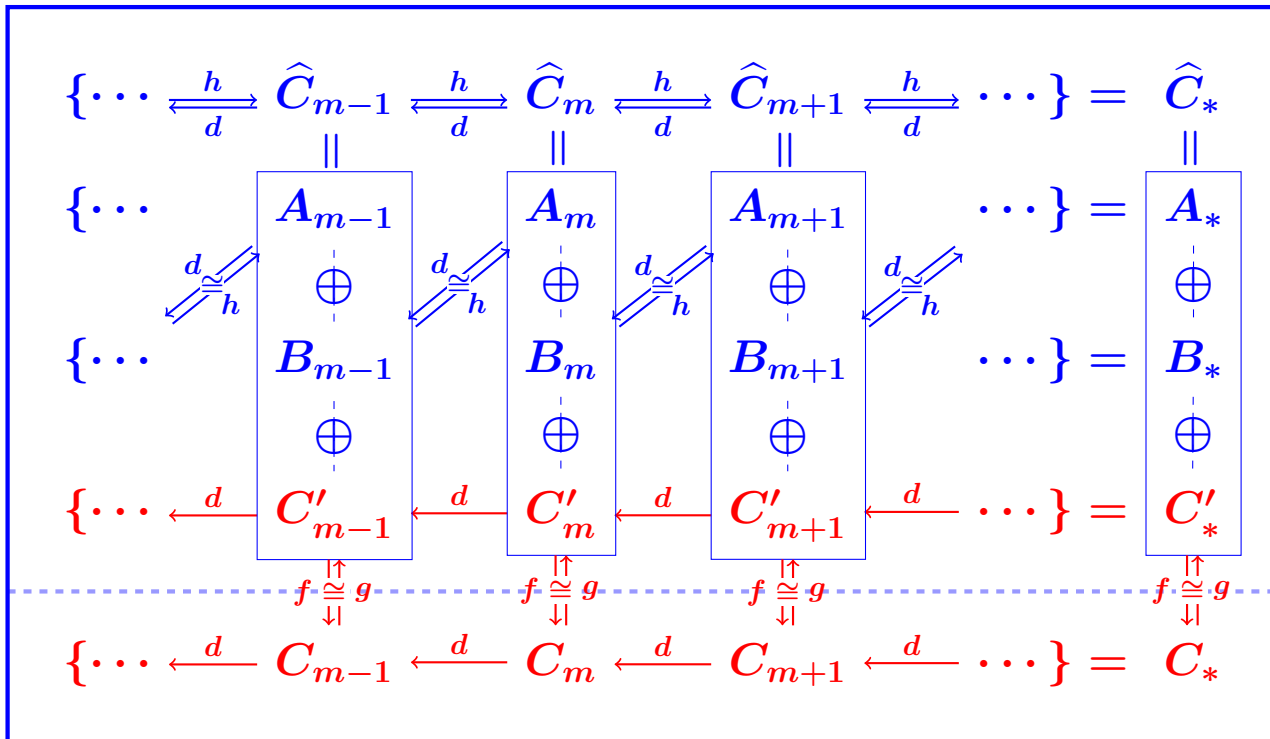
Theorem: The BPL determines a new reduction:

$$\rho' : h + \delta_h \circlearrowleft (\widehat{C}_*, \widehat{d}_* + \widehat{\delta}_*) \xrightleftharpoons[f + \delta_f]{g + \delta_g} (C_*, d_* + \delta_{d_*})$$

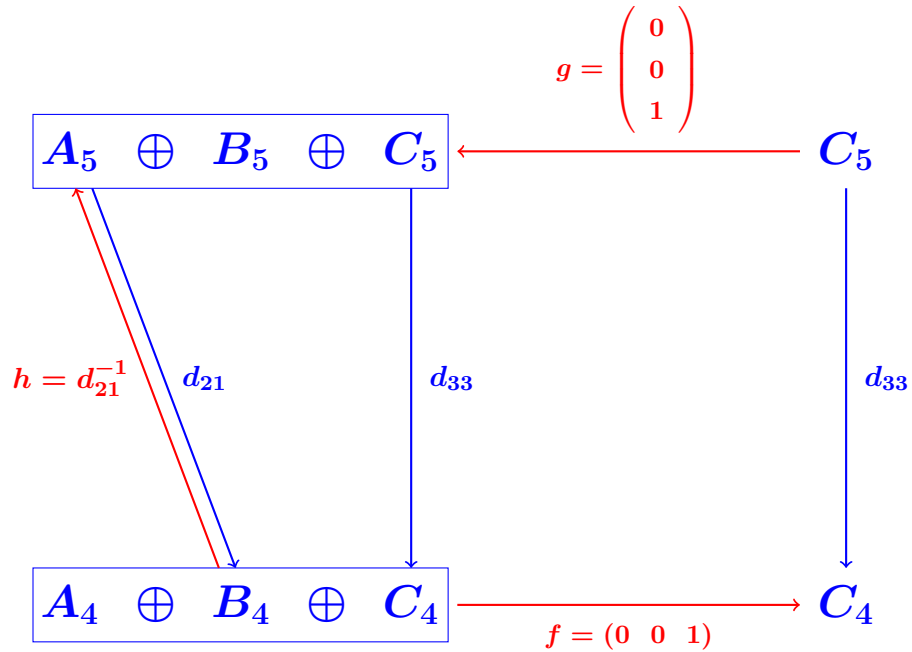


Proof:

Reduction Diagram:

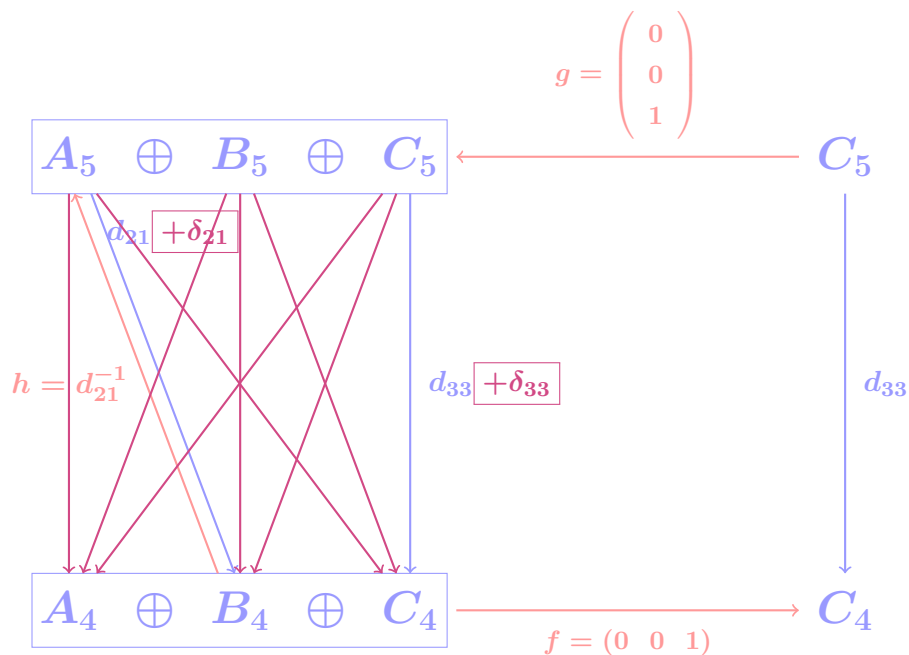


Main part:



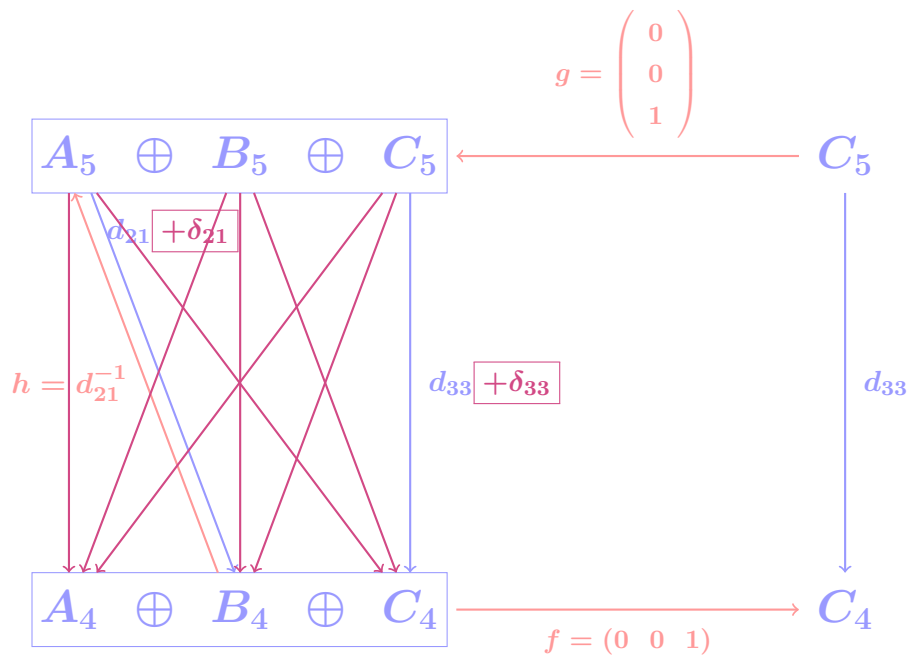
with  $d_{21} = \text{isomorphism}$ .

$$\text{Perturbation} = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} :$$



Question:  $(d_{21} + \delta_{21})$  again isomorphism?

(applying the **Global Hexagonal Theorem** possible ?)



But  $d_{21}$  invertible with  $d_{21}h = 1 \Rightarrow$

$$d_{21} + \delta_{21} = d_{21} + d_{21}h\delta_{21} = d_{21}(1 + h\delta_{21})$$

$\Rightarrow d_{21} + \delta_{21}$  invertible  $\Leftrightarrow (1 + h\delta_{21})$  invertible.

A sufficient condition is  $h\delta_{21}$  nilpotent, in which case:

$$(1 + h\delta_{21})^{-1} = \sum_{i=0}^{\infty} (-1)^i (h\delta_{21})^i$$

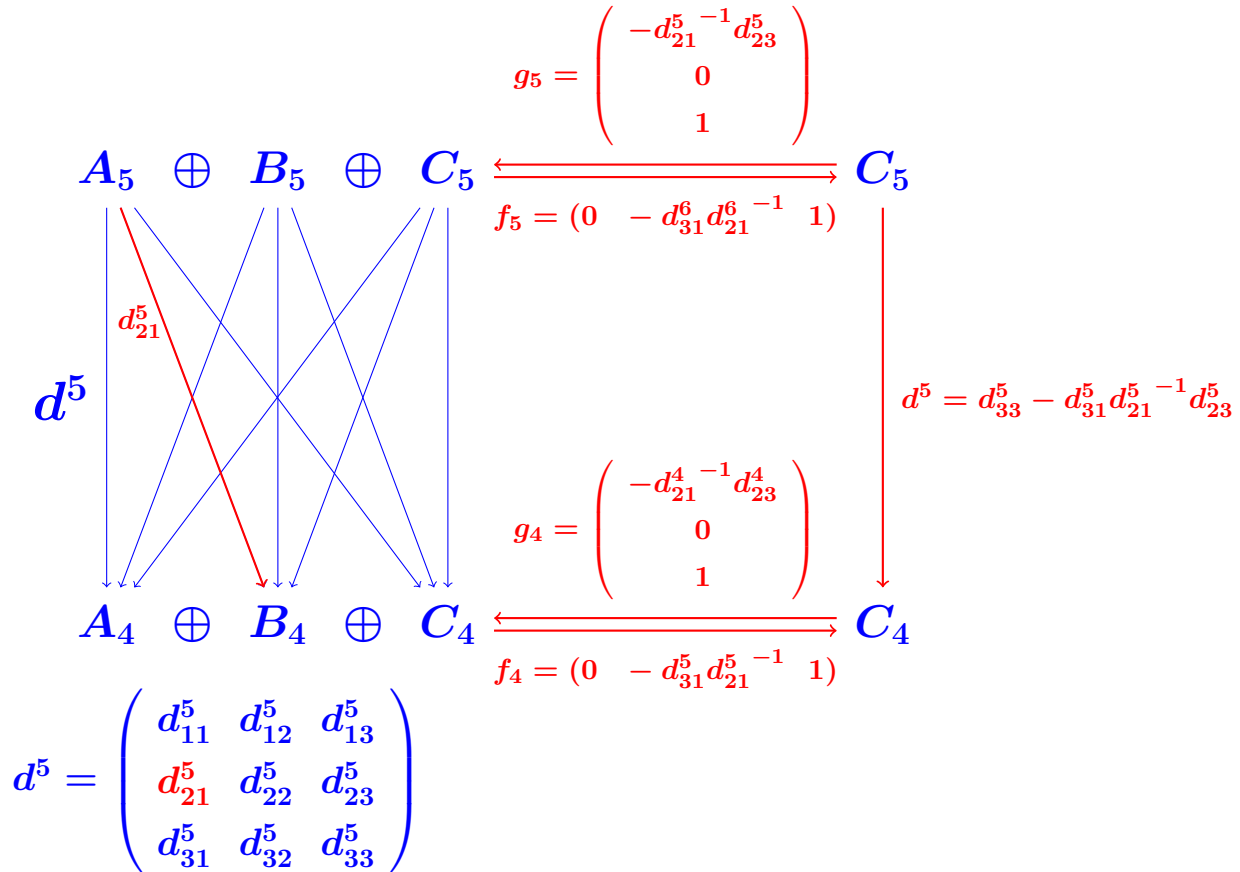
Then:

$$(d_{21} + \delta_{21})^{-1} =: h' := \left( \sum_{i=0}^{\infty} (-1)^i (h\delta_{21})^i \right) h$$

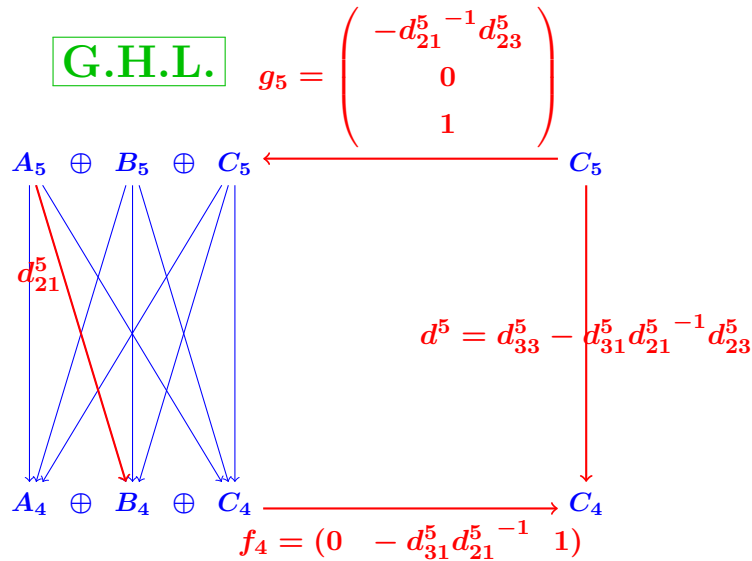
Remark:

$$\left( \sum_{i=0}^{\infty} (-1)^i (h\delta_{21})^i \right) h = \left( \sum_{i=0}^{\infty} (-1)^i (h\delta)^i \right) h$$

# Global Hexagonal Theorem:



Applying to our situation:

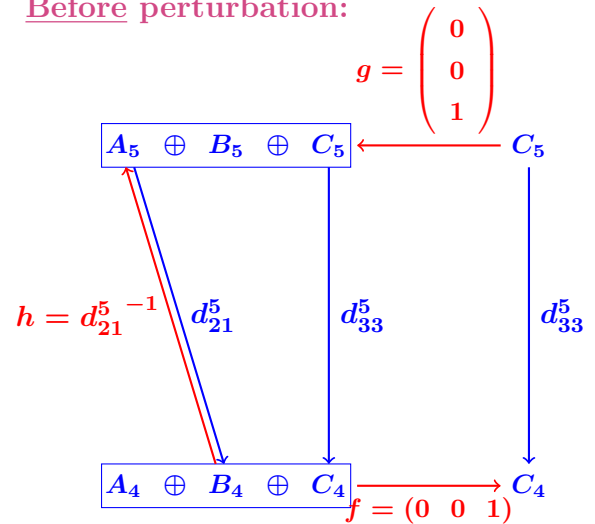


$$d_{21}^5 \mapsto d_{21}^5 + \delta_{21}^5$$

$$d_{21}^{5-1} \mapsto h' = \left( \sum_{i=0}^{\infty} (-1)^i (h\delta)^i \right) h$$

= Homological Perturbation Theorem

Before perturbation:



$$g_5 \mapsto (1 - h'\delta)g$$

$$f_4 \mapsto f(1 - \delta h')$$

$$d^5 \mapsto (d_{33}^5 + \delta_{33}^5) - f\delta h'\delta g$$

$$= d_{33}^5 + f\delta g - f\delta h'\delta g$$

QED

6/6. The **topological** case.

Corollary: The **HPT** can easily be **extended**  
to **topological** situations.

Example 1: **Banach** situations:

$$\|h\delta_{21}\| < 1 \Rightarrow (1 + h\delta_{21}) \text{ invertible} \Rightarrow \text{OK.}$$

Example 2: **Frechet** situations:

The **Nash-Moser-Schwartz** technology

often allows to prove  $(1 - h\delta_{21})$  is **invertible**  $\Rightarrow$  **OK**.



The END

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