

Spectral sequences for computing persistent homology

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(Joint work with J. Heras, J. Rubio, and F. Sergeraert)

Mathematics, Algorithms and Proofs
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Spectral sequences

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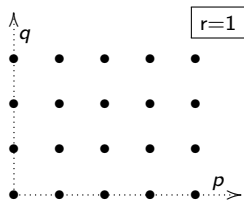
Definition

A *spectral sequence* $E = (E^r, d^r)_{r \geq 1}$ is a family of bigraded \mathbb{Z} -modules $E^r = \{E_{p,q}^r\}$, each provided with a differential $d^r = \{d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r\}$ of bidegree $(-r, r-1)$ and with isomorphisms $H(E^r, d^r) \cong E^{r+1}$ for every $r \geq 1$.

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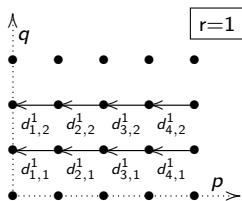
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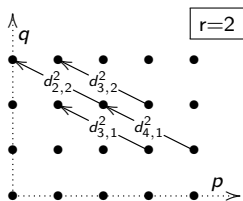
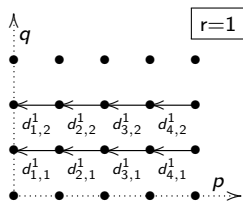
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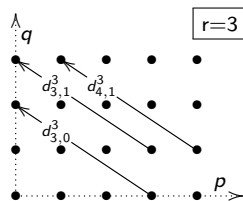
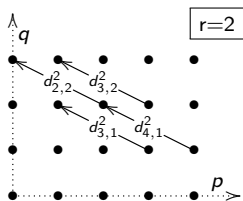
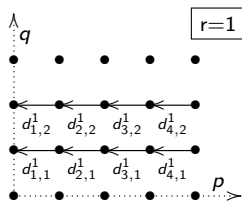
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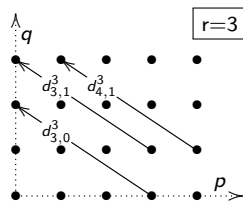
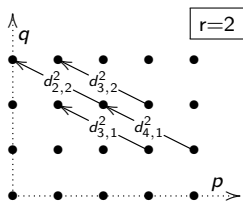
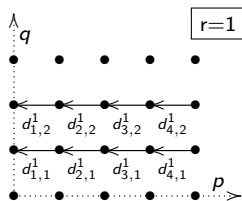
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Since $E_{p,q}^{r+1}$ is a subquotient of $E_{p,q}^r$ for each $r \geq 1$, one can define the *final groups* of the spectral sequence as $E_{p,q}^\infty = \bigcap_{r \geq 1} E_{p,q}^r$.

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- The *Eilenberg-Moore spectral sequence* converges to the homology groups of the loop space of a simplicial set.
- The *Adams spectral sequence* converges to the homotopy groups of a simplicial set X .

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Theorem (Serre spectral sequence)

Let $G \hookrightarrow E \rightarrow B$ be a fibration with a simply connected base space B . Then a first quadrant spectral sequence $E = (E^r, d^r)_{r \geq 1}$ can be defined with $E_{p,q}^2 = H_p(B, H_q(G))$ and $E^1 \Rightarrow H_*(E)$.

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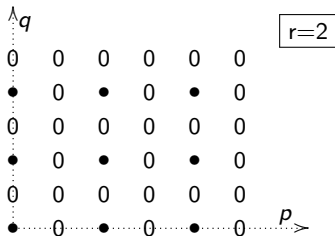
Suppose $H_i(G)$ and $H_i(B)$ are zero for odd i and free abelian for even i . The entries $E_{p,q}^2$ of the E^2 page are then zero unless p and q are even.

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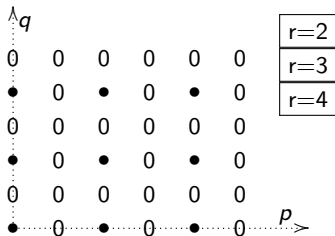


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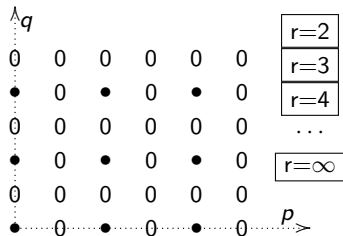


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Example: $K(\mathbb{Z}, 2) \hookrightarrow X \rightarrow S^3$, where $H_{2i}(K(\mathbb{Z}, 2)) = \mathbb{Z}$ and $H_{2i+1}(K(\mathbb{Z}, 2)) = 0$ for all i .

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$$\begin{array}{cccccc}
 \uparrow q & & & & & \\
 \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \\
 \vdots & & & & & \\
 0 & 0 & 0 & 0 & 0 & \\
 \vdots & & & & & \\
 \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \\
 \vdots & & & & & \\
 0 & 0 & 0 & 0 & 0 & \\
 \vdots & & & & & \\
 \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \\
 \vdots & & & & & \\
 0 & 0 & 0 & 0 & 0 & \\
 \vdots & & & & & \\
 \mathbb{Z} & \cdots & 0 & \cdots & \mathbb{Z} & \cdots & 0 \cdots p \rightarrow
 \end{array}
 \quad
 \begin{array}{|c|}
 \hline
 r = 2 \\
 \hline
 r = 3 \\
 \hline
 \end{array}$$

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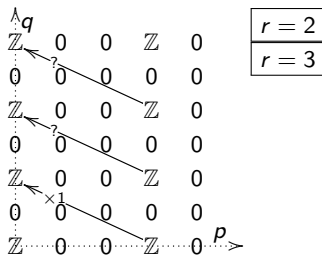
$$\begin{array}{cccccc}
 \uparrow q & & & & & \\
 \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \\
 \vdots & & & & & \\
 0 & 0 & 0 & 0 & 0 & \\
 \vdots & & & & & \\
 \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \\
 \vdots & & & & & \\
 0 & 0 & 0 & 0 & 0 & \\
 \vdots & & & & & \\
 \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \\
 \vdots & & & & & \\
 0 & 0 & 0 & 0 & 0 & \\
 \vdots & & & & & \\
 \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \rightarrow p
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$$\begin{array}{ccccccc}
 \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & & \\
 \vdots & \swarrow \times 3 & & & & & \\
 \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & & \\
 \vdots & \swarrow \times 2 & & & & & \\
 \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & & \\
 \vdots & \swarrow \times 1 & & & & & \\
 \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & & \\
 \vdots & & & & & & \\
 \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \xrightarrow{p} &
 \end{array}$$

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 \hline
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 \hline
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 \mathbb{Z} & 0 & 0 & 0 & 0 & & \\
 \vdots & & & & & & \\
 \mathbb{Z}_3 & 0 & 0 & 0 & 0 & & \\
 \vdots & & & & & & \\
 \mathbb{Z}_2 & 0 & 0 & 0 & 0 & & \\
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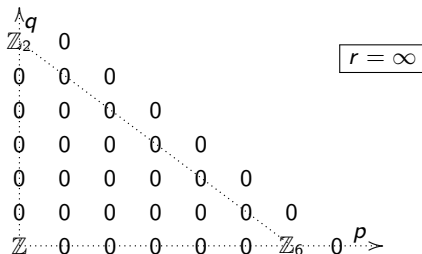
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$$\begin{array}{cccccccc}
 & \overset{q}{\Lambda} & & & & & & & \\
 \mathbb{Z}_2 & 0 & & & & & & & \\
 \vdots & 0 & 0 & & & & & & \\
 \vdots & 0 & 0 & 0 & & & & & \\
 \vdots & 0 & 0 & 0 & 0 & & & & \\
 \vdots & 0 & 0 & 0 & 0 & 0 & & & \\
 \vdots & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 \mathbb{Z} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z}_6 & 0 \xrightarrow{p}
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Spectral sequence of a filtered complex

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Theorem

Let C be a chain complex with a filtration. There exists a spectral sequence with

$$E_{p,q}^r = \frac{Z_{p,q}^r + C_{p+q}^{p-1}}{d_{p+q+1}(Z_{p+r-1,q-r+2}^{r-1}) + C_{p+q}^{p-1}}$$

where $Z_{p,q}^r$ is $Z_{p,q}^r = \{a \in C_{p+q}^p \mid d_{p+q}(a) \in C_{p+q-1}^{p-r}\} \subseteq C_{p+q}^p$, and

$d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ is the morphism induced by

$d_{p+q} : C_{p+q} \rightarrow C_{p+q-1}$. This spectral sequence converges to $H_*(C)$, that is, there are natural isomorphisms

$$E_{p,q}^\infty \cong \frac{H_{p+q}^p(C)}{H_{p+q}^{p-1}(C)}$$

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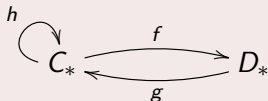
Problem: the groups $E_{p,q}^r$ can only be determined in very simple (finite) situations.

Effective homology

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Definition

A *reduction* ρ between two chain complexes C_* and D_* (denoted by $\rho : C_* \rightrightarrows D_*$) is a triple $\rho = (f, g, h)$



satisfying the following relations:

- 1) $fg = \text{Id}_{D_*}$;
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$$\begin{array}{ccc}
 & & \\
 & \curvearrowright & \\
 & h & \\
 & \downarrow & \\
 C_* & \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} & D_*
 \end{array}$$

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If $C_* \rightrightarrows D_*$, then $C_* \cong D_* \oplus A_*$, with A_* acyclic, which implies that $H_n(C_*) \cong H_n(D_*)$ for all n .

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$$\begin{array}{ccc}
 & B_* & \\
 \swarrow & & \searrow \\
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 \end{array}$$

$$\begin{array}{ccc}
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This implies that $H_n(X) \cong H_n(HC_*)$ for all n .

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- Otherwise, the program uses the effective homology.

Example:

$$X = \Omega(\Omega(\Omega(P^\infty\mathbb{R}/P^3\mathbb{R}) \cup_4 D^4) \cup_2 D^2)$$

$$H_5(X) = \mathbb{Z}_2^{23} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{16}$$

$$H_6(X) = \mathbb{Z}_2^{52} \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}^3$$

$$H_7(X) = \mathbb{Z}_2^{113} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8^3 \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{32} \oplus \mathbb{Z}$$

Algorithm computing spectral seq. of filtered complexes

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However, the effective homology method allows us to determine, as a by-product, this kind of spectral sequences, obtaining a real algorithm.

This algorithm has been implemented as a set of programs (about 2500 lines) enhancing the Kenzo system, that determine the groups $E_{p,q}^r$ and also the differential maps $d_{p,q}^r$ for every level r .

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- If the filtered chain complex (C_*, F_C) is not effective, but with **effective homology** $C_* \Leftarrow D_* \Rightarrow HC_*$, then appropriate filtrations of D_* and HC_* can also produce the spectral sequence.

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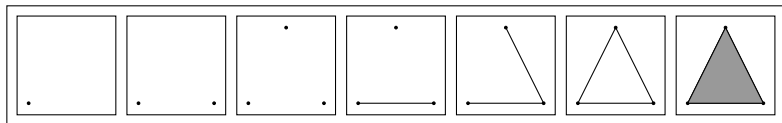
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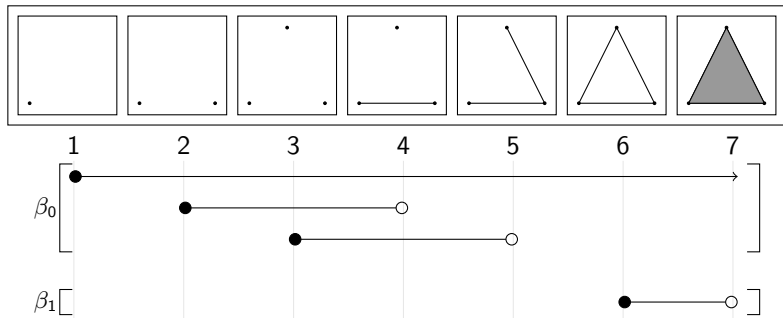
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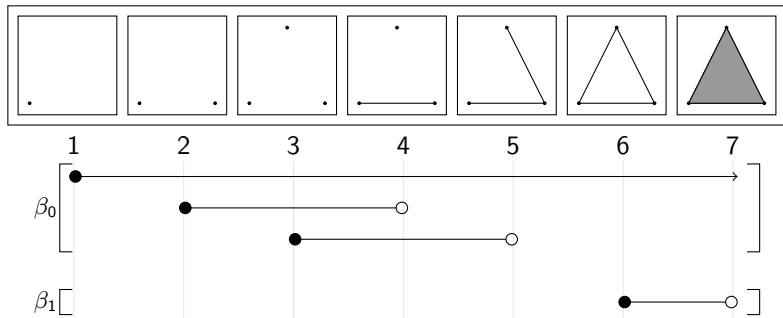
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Roughly speaking:

- Homology detects topological features (connected components, holes, and so on).
- Persistent homology describes the evolution of topological features looking at consecutive thresholds.

Integral persistent homology

Integral persistent homology

We generalize the definition of persistent homology to the case of coefficients over \mathbb{Z} by means of the groups:

$$H_n^{i,j,k} = H_n^{i,j} \cap (f_n^{j,k})^{-1}(H_n^{i-1,k}) \subseteq H_n^j$$

which define a filtration between $H_n^{i-1,j}$ and $H_n^{i,j}$:

$$H_n^{i-1,j} = H_n^{i,j,j} \subseteq H_n^{i,j,j+1} \subseteq H_n^{i,j,j+2} \subseteq \dots \subseteq H_n^{i,j,m} \subseteq H_n^{i,j}$$

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Then one can define the quotients:

$$BD_n^{i,k} = \frac{H_n^{i,i,k}}{H_n^{i,i,k-1}} = \frac{H_n^{i,i+1,k}}{H_n^{i,i+1,k-1}} = \dots = \frac{H_n^{i,k-2,k}}{H_n^{i,k-2,k-1}} = \frac{H_n^{i,k-1,k}}{H_n^{i-1,k-1}}$$

The notation $BD_n^{i,k}$ should be read as *the group of homological classes born at time i and dying at time k* , in fact a group of equivalence classes modulo inferior homology groups.

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When working over a field, the groups $BD_n^{i,k}$ are uniquely determined (up to isomorphism) by their rank, $\mu_n^{i,k}$:

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In this way, the classical definition of *barcode diagram* is not valid for the integer case.

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The so called “Spectral sequence theorem” in the book *Computational topology: an introduction* by H. Edelsbrunner and J. Harer claims that:

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The total rank of the groups of dimension $p + q$ in the level $r \geq 1$ of the associated spectral sequence equals the number of points in the $(p + q)$ -th persistence diagram whose persistence is r or larger, that is,

$$\sum_{p=1}^m \text{rank } E_{p,q}^r = \text{card}\{a \in \text{Dgm}_{p+q}(f) \mid \text{pers}(a) \geq r\}$$

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However, the formula is erroneous because in the left side there can be more elements than in the right side; the formula should be therefore an inequality.

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Theorem

Let $K^1 \subseteq K^2 \subseteq \dots \subseteq K^m = K$ be a filtered simplicial complex. For each dimension n and for each $0 \leq i < k \leq m$, one has a canonical isomorphism

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As a corollary, they include a corrected version of the erroneous theorem by H. Edelsbrunner and J. Harer.

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- In the integer case, one can find extension problems in order to determine $H_n^{i,j}$ and the *intermediate* groups $H_n^{i,j,k}$.

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In fact a small modification of our programs for spectral sequences is sufficient to determine the groups $BD_n^{i,k}$, which can be expressed as a quotient:

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$$H_n^{i,j} = \frac{Z_{i,n-i}^i}{d_{n+1}(Z_{j,n-j+1}^{j-i})}$$

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- They can also be applied in the infinite case, where the effective homology method can be used to determine the groups $H_n^{i,j}$ by means of an *equivalence* between the initial chain complex C and an auxiliary chain complex of finite type.

Integer persistence of torus and Klein bottle

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We use the program *JavaPlex* to create an explicit metric space for 1000 random points on a flat torus (respectively on a flat Klein bottle) and then construct an associated *lazy witness* filtered simplicial complex.

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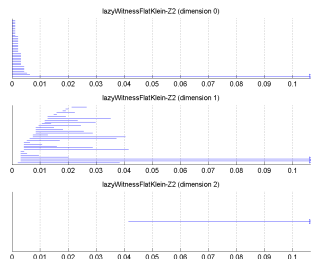
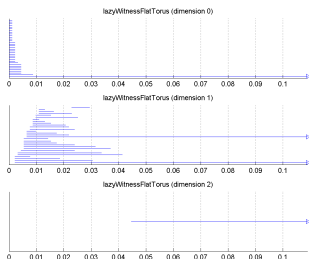
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Persistent Homology  $BD^{\{0,3\}}_0$ 
Component Z
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> (prst-hmlg-group torus 0 4 0)
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```
Persistent Homology  $BD^{\{0,4\}}_0$ 
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Component Z
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Component  $\mathbb{Z}$ 
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The space X_3 is a twisted Cartesian product $X_3 = K(\mathbb{Z}_2, 3) \times_{k_3} K(\mathbb{Z}_2, 2)$, total space of a fibration $K(\mathbb{Z}_2, 3) \hookrightarrow X_3 \rightarrow K(\mathbb{Z}_2, 2)$.

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The space X_3 is a twisted Cartesian product $X_3 = K(\mathbb{Z}_2, 3) \times_{k_3} K(\mathbb{Z}_2, 2)$, total space of a fibration $K(\mathbb{Z}_2, 3) \hookrightarrow X_3 \rightarrow K(\mathbb{Z}_2, 2)$. The object X3 is already of finite type, but its effective homology gives us an associated effective chain complex which is much smaller.

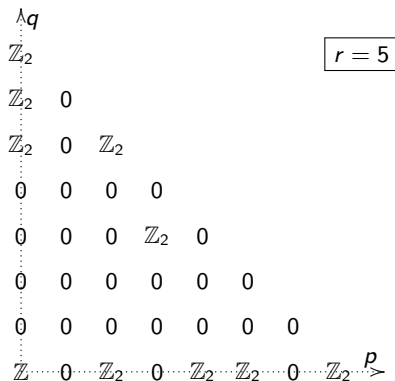
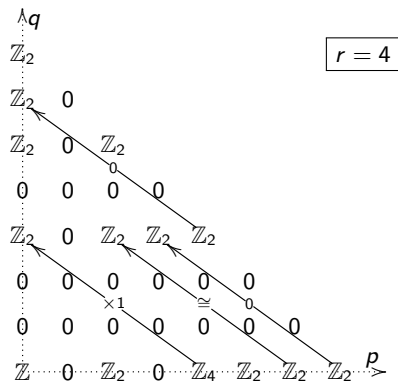
Integer persistence of Postnikov towers

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This corresponds to the image of the differential $d_{4,0}^4 : E_{4,0}^4 \rightarrow E_{0,3}^4$ in the spectral sequence.

Integer persistence of Postnikov towers

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```
> (prst-hmlg-group X3 0 8 7)
Persistent Homology  $BD^{\{0,8\}}_7$ 
Component  $\mathbb{Z}/2\mathbb{Z}$ 
> (prst-hmlg-group X3 2 8 7)
Persistent Homology  $BD^{\{2,8\}}_7$ 
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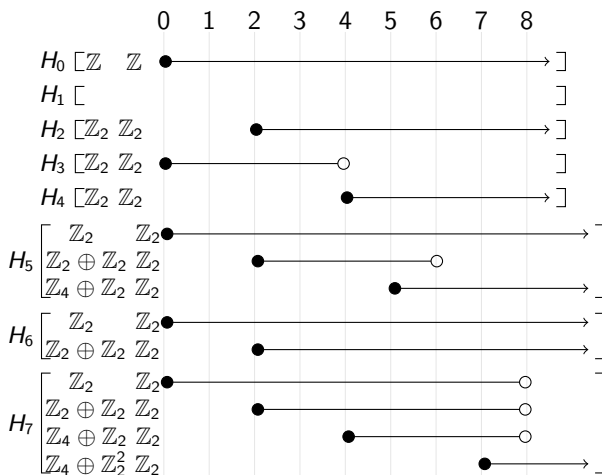
```
> (prst-hmlg-group X3 0 8 7)
Persistent Homology  $BD^{\{0,8\}}_7$ 
Component  $\mathbb{Z}/2\mathbb{Z}$ 
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> (total-prst-hmlg-group X3 4 7 7)
Persistent Homology  $H^{\{4,7\}}_7$ 
Component  $\mathbb{Z}/2\mathbb{Z}$ 
Component  $\mathbb{Z}/4\mathbb{Z}$ 
```

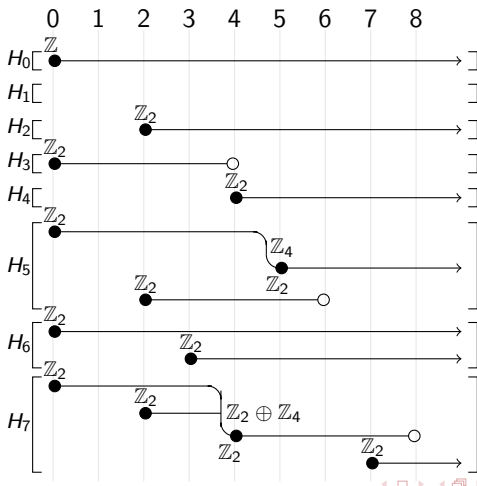
Integer persistence of Postnikov towers

Integer barcode diagram of the space $X_3 = K(\mathbb{Z}_2, 3) \times_{k_3} K(\mathbb{Z}_2, 2)$.



Integer persistence of Postnikov towers

Alternative description of the integer barcode diagram of $X_3 = K(\mathbb{Z}_2, 3) \times_{k_3} K(\mathbb{Z}_2, 2)$.



Conclusions and further work

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- As a by-product, our persistent homology programs can be applied also over spaces of infinite nature.

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- For big digital images, discrete vector fields can be used to construct a *reduced* (effective) chain complex.