

Matroids over a ring: motivations, examples, applications

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MAP 2014

The partition function

Let be $X = \{a_1, \dots, a_n\} \subseteq \mathbb{Z}_{>0}$.

Problem

In how many ways can we write $\lambda \in \mathbb{N}$ as

$$\lambda = \sum_{i=1}^n x_i a_i, \quad x_i \in \mathbb{N}?$$

Fixed X , this is a function of λ , which we denote by $\mathcal{P}_X(\lambda)$ and we call the **partition function**.

Example

A cashpoint has only 20 and 50 euros banknotes. In how many ways an amount of λ euros can be paid?

Here $X = \{20, 50\}$. $\mathcal{P}_X(\lambda)$ counts the number of solutions of the equation

$$20x + 50y = \lambda, \quad x, y \in \mathbb{N}.$$

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Example $X = \{20, 50\}$

The solution of this problem depends on the coset of λ :

$$\mathcal{P}_X(\lambda) = \begin{cases} 0 & \text{if 10 does not divide } \lambda \\ \frac{\lambda+100}{100} & \text{if } \lambda \equiv 0 \pmod{100} \\ \frac{\lambda-10}{100} & \text{if } \lambda \equiv 10 \pmod{100} \\ \frac{\lambda+80}{100} & \text{if } \lambda \equiv 20 \pmod{100} \\ \dots & \\ \frac{\lambda+10}{100} & \text{if } \lambda \equiv 90 \pmod{100} \end{cases}$$

Then on every coset of $100\mathbb{Z} \subseteq \mathbb{Z}$, $\mathcal{P}_X(\lambda)$ is a polynomial.

We call such a function a quasi-polynomial.

(More generally, a function $q : \mathbb{Z}^d \rightarrow \mathbb{C}$ is a **quasi-polynomial** if there is a finite index subgroup of \mathbb{Z}^d such that the restriction of q to every coset is polynomial).

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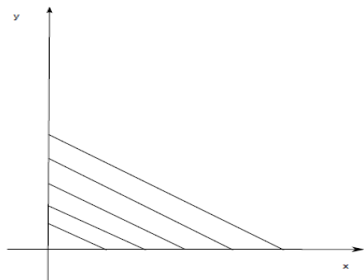
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Example $X = \{20, 50\}$

Geometrically, the conditions $20x + 50y = \lambda$, $x, y \geq 0$ define a variable segment $P_X(\lambda)$ in \mathbb{R}^2 .

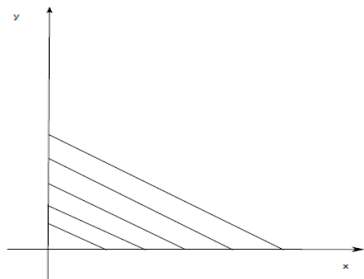


We count integer points in this segment:

$$P_X(\lambda) = |P_X(\lambda) \cap \mathbb{Z}^2|.$$

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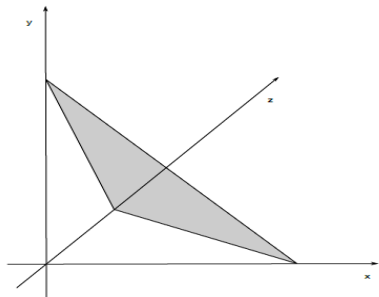
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Example $X' = \{20, 50, 100\}$

Let $X' = \{20, 50, 100\}$. Then we have the equation

$$20x + 50y + 100z = \lambda, \quad x, y, z \geq 0$$

defining a variable triangle $P_{X'}(\lambda)$ in \mathbb{R}^3 , obtained by intersecting the positive octant of \mathbb{R}^3 with a plane.



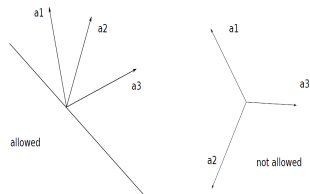
Again, $\mathcal{P}_{X'}(\lambda)$ is the number of integer points in $P_{X'}(\lambda)$.

Vector partition function

Let now $X = \{a_1, \dots, a_n\} \subseteq \Gamma \simeq \mathbb{Z}^d$. For every $\lambda \in \Gamma$, we define $\mathcal{P}_X(\lambda)$ as the number of ways we can write

$$\lambda = \sum_{i=1}^n x_i a_i \quad x_i \in \mathbb{N}.$$

Since we want this number to be finite, we require that all the a_i lie on the same side of a hyperplane:



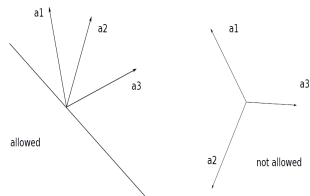
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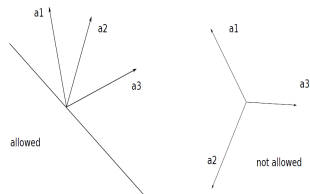
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A variable polytope

The equation $\lambda = \sum_{i=1}^n x_i a_i$ is indeed a system of equations (one for every coordinate of λ), that identifies a subspace of \mathbb{R}^n .

The intersection of this subspace with the positive orthant is a **variable polytope** $P_X(\lambda)$, and

$$\mathcal{P}_X(\lambda) = |P_X(\lambda) \cap \mathbb{Z}^n|.$$

Then, the partition function $\mathcal{P}_X(\underline{\lambda})$ is related with another function, defined as the **volume** of the same polytope, and called the **multivariate spline**:

$$\mathcal{S}_X(\underline{\lambda}) = \text{vol}(P_X(\underline{\lambda})).$$

In fact, \mathcal{S}_X is “piecewise polynomial”, while \mathcal{P}_X is “piecewise quasi-polynomial”.

The “local pieces” generate the so-called “Dahmen-Micchelli modules”, which have interesting algebraic properties and geometric realizations [De Concini-Procesi-Vergne].

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Relation with arrangements

De Concini-Procesi's approach: applying "Laplace transform" L we get

$$L\mathcal{S}_X = \prod_{i=1}^h \frac{1}{a_i} \qquad L\mathcal{P}_X = \prod_{i=1}^h \frac{1}{1 - e^{2\pi i a_i}}$$

where we view every a_i as a linear functional on the dual space.

Notice that $L\mathcal{S}_X$ is rational function defined on the complement of a **hyperplane arrangement** \mathcal{H}_X .

Similarly, $L\mathcal{P}_X$ is defined on the complement of a **toric arrangement** \mathcal{T}_X .

The geometry of these arrangements encodes much information on the functions \mathcal{S}_X and \mathcal{P}_X .

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An example of arrangements

Take $V = \mathbb{R}^2$ with coordinates (x, y) , $T = (\mathbb{S}^1)^2$ with coordinates (t, s) ,
and

$$X = \{(2, 0), (0, 3), (-1, 1)\} \subset \mathbb{Z}^2.$$

We associate to X three objects:

- 1 a finite hyperplane arrangement \mathcal{H}_X given in V by the equations

$$2x = 0, 3y = 0, -x + y = 0;$$

- 2 a periodic hyperplane arrangement \mathcal{A}_X given in V by the conditions

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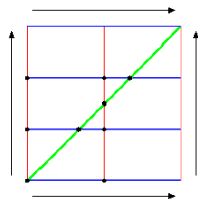
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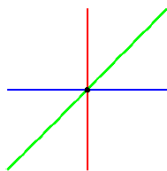
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Hyperplane vs toric arrangements

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toric arrangement



hyperplane arrangement

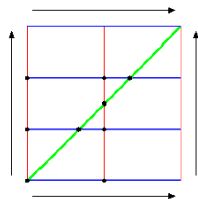
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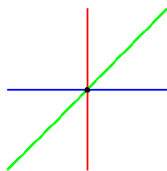
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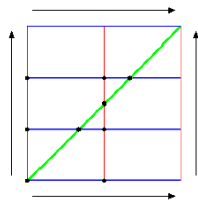
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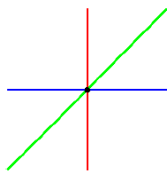
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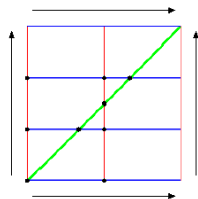
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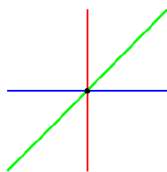
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Matroids: definition

A structure that retains the linear algebraic information of a list of vector already exists since the 30s: **matroids** [Whitney, Maclane].

It has many apparently unrelated definitions. (Rota: “cryptomorphism”.)

Definition

A **matroid** M on the finite *ground set* E assigns to each subset $A \subseteq E$ a rank $\text{rk}(A) \in \mathbb{Z}_{\geq 0}$, such that: [...]

Main example: **realizable** matroids

Let v_1, \dots, v_n be vectors in a vector space V .

$$\text{rk}(A) := \dim \text{span}\{v_i : i \in A\}$$

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Main example: **realizable** matroids

Let v_1, \dots, v_n be vectors in a vector space V .

$$\text{rk}(A) := \dim \text{span}\{v_i : i \in A\}$$

Matroids: definition

A structure that retains the linear algebraic information of a list of vector already exists since the 30s: **matroids** [Whitney, Maclane].

It has many apparently unrelated definitions. (Rota: “cryptomorphism”.)

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$$(0) \text{rk}(\emptyset) = 0$$

$$(1) \text{rk}(A) \leq \text{rk}(A \cup \{b\}) \leq \text{rk}(A) + 1 \quad \forall A \not\ni b$$

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Example

$X = \{(2, 0), (0, 3), (1, -1)\}$	A	\emptyset	1	2	3	13	12	23	123
	$\text{rk}(A)$	0	1	1	1	2	2	2	2

Enriched variants of matroids

You might want to capture more than just the linear dependences of your vectors:

- **Valuated matroids** come from configurations over a *field with valuation*, and remember valuations. [Dress-Wenzel]
- **Arithmetic matroids** come from configurations over \mathbb{Z} , and remember indices of sublattices. [D'Adderio-M.]

Matroids over rings encompass these constructions, by taking a new approach: not a matroid decorated with extra data, but a theory with only *one* simple, algebraic axiom.

The notion of a matroid over a ring R axiomatizes dependencies of elements in an R -module, where R is any commutative ring.

So our theory aims to generalize matroid theory, in the same sense as commutative algebra generalizes linear algebra.

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Matroids over \mathbb{Z} : an example

Let v_1, \dots, v_n be a configuration of vectors in an R -module N .

Already in the case $R = \mathbb{Z}$ we see that it is convenient to take a system of axioms for the *quotients* $N/\langle v_i | i \in A \rangle$:

Realizable example

$X = \{(2, 0), (0, 3), (1, -1)\}$	A	\emptyset	1	2	12
	$M(A)$	\mathbb{Z}^2	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/3$	$\mathbb{Z}/6$
	A	3	13	23	123
	$M(A)$	\mathbb{Z}	$\mathbb{Z}/2$	$\mathbb{Z}/3$	0

Definition

Let R be a commutative ring and E be a finite set.

A **matroid over R** on the *ground set* E is a function M assigning to each subset $A \subseteq E$ a finitely-generated R -module $M(A)$ satisfying the following **axiom**:

for all $A \subseteq E$ and $b, c \notin A$, there are elements

$$x = x(b, c), \quad y = y(b, c) \in M(A)$$

such that there is a diagram

$$\begin{array}{ccc} M(A) & \xrightarrow{/x} & M(A \cup \{b\}) \\ /y \downarrow & \lrcorner & \downarrow /y \\ M(A \cup \{c\}) & \xrightarrow{/x} & M(A \cup \{b, c\}). \end{array}$$

In other words, for all $A \subseteq E$ and $b \neq c \notin A$, the diagram above is a pushout square in which all the arrows are surjections with cyclic kernel.

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Realizability

Fundamental example: “vector configurations” in an R -module.

Given a f.g. R -module N and a list $X = x_1, \dots, x_n$ of elements of N , we have a matroid M_X associating to $A \subseteq X$ the quotient

$$M_X(A) = N / \left(\sum_{x \in A} Rx \right).$$

For each $x_i \in X$ there is a quotient map

$$M_X(A) \xrightarrow{/x_i} M_X(A \cup \{x_i\})$$

and this system of maps obviously satisfies the axiom.

We say that a matroid M over R is **realizable** if it actually comes from such a list.

Of course not all matroids over R are realizable!

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Classical matroids are matroids over fields

We can, and will, assume that the module $M(E)$ has no nontrivial projective summands, since this makes many results simpler to state.

Theorem 1 (Fink-M.)

Matroids over a field \mathbb{K} are equivalent to (classical) matroids.

A f.g. \mathbb{K} -module is determined by its *dimension* $\in \mathbb{Z}$.

If v_1, \dots, v_n are vectors in \mathbb{K}^r ,
the dimension of $\mathbb{K}^r / \langle v_i : i \in N \rangle$ is $r - \text{rk}(A)$, the **corank** of A .

Example

$$X = \{(2, 0), (0, 3), (1, -1)\}$$

A	\emptyset	1	2	12	3	13	23	123
$M(A)$	\mathbb{R}^2	\mathbb{R}	\mathbb{R}	0	\mathbb{R}	0	0	0

Note: The definition of matroids over \mathbb{K} is blind to which field \mathbb{K} is, but for *realizability* the choice of \mathbb{K} matters.

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Sum, deletion, contraction, duality

Let M and M' be matroids over R on E and E' .

We define their **direct sum** $M \oplus M'$ on $E \amalg E'$ by

$$(M \oplus M')(A \amalg A') = M(A) \oplus M'(A').$$

For $i \in E$, we define two matroids over R on the ground set $E \setminus \{i\}$: the **deletion** of i in M , denoted $M \setminus i$, by

$$(M \setminus i)(A) = M(A)$$

and the **contraction** of i in M , denoted M / i , by

$$(M / i)(A) = M(A \cup \{i\}).$$

When R is a Dedekind domain, we can also define a **dual matroid** M^* having the expected properties (omitted).

If M is realizable, $M \setminus i$ and M / i can be realized in the usual way, while M^* can be realized by a generalization of Gale duality.

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Tensor product. Localizations and generic matroid

Let $R \rightarrow S$ be a map of rings. Then the tensor product $— \otimes_R S$ is a functor $R\text{-Mod} \rightarrow S\text{-Mod}$. If M is a matroid over R , then

$$(M \otimes_R S)(A) \doteq M(A) \otimes_R S.$$

defines a matroid over S .

Two special cases will be fundamental for us:

- 1 For every prime ideal \mathfrak{m} of R , let $R_{\mathfrak{m}}$ be the localization of R at \mathfrak{m} . We call $M \otimes_R R_{\mathfrak{m}}$ the **localization** of M at \mathfrak{m} .
- 2 If R is a domain, let $\text{Frac}(R)$ be the fraction field of R . Then we call $M \otimes_R \text{Frac}(R)$ the **generic matroid** of M .

Notice that every matroid over $R_{\mathfrak{m}}$ induces a matroid over the residue field $R_{\mathfrak{m}}/(\mathfrak{m})$.

We can study the matroid M via all these “classical” matroids.

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Dedekind rings and DVR

From now on, we will always assume R to be a **Dedekind domain** (i.e., an integral domain in which every nonzero proper ideal factors into a product of prime ideals).

The localization of a Dedekind domain at a prime ideal is a DVR (i.e. a Dedekind domain that is not a field and has a unique maximal ideal \mathfrak{m}). Any indecomposable f.g. module over a DVR R is isomorphic to either R or R/\mathfrak{m}^n for some integer $n \geq 1$.

So a f.g. R -module are parametrized by “partitions” that may have some infinitely long lines.

We denote by t_n the cardinality of the n -truncation of such a “partition”. Our first result is a combinatorial characterization of matroids over a DVR:

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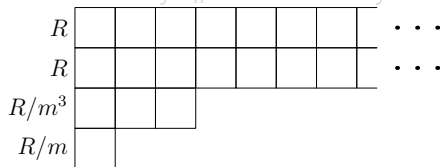
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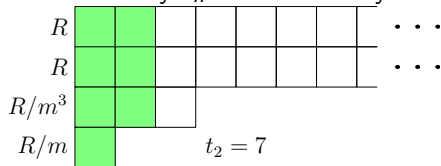
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From now on, we will always assume R to be a **Dedekind domain** (i.e., an integral domain in which every nonzero proper ideal factors into a product of prime ideals).

The localization of a Dedekind domain at a prime ideal is a DVR (i.e. a Dedekind domain that is not a field and has a unique maximal ideal \mathfrak{m}). Any indecomposable f.g. module over a DVR R is isomorphic to either R or R/\mathfrak{m}^n for some integer $n \geq 1$.

So a f.g. R -module are parametrized by “partitions” that may have some infinitely long lines.

We denote by t_n the cardinality of the n -truncation of such a “partition”.



Our first result is a combinatorial characterization of matroids over a DVR:

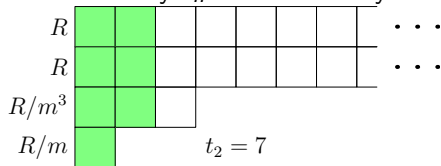
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Local theory: matroids over a DVR

Theorem (Fink, M.)

$M : 2^E \rightarrow \{f. g. R\text{-modules}\}$ is a matroid over R if and only if:
for every 1-element minor $M(A) \rightarrow M(A \cup b)$ the difference of the two “partitions” is a (Pieri-like) stripe,
and on every 2-element minor the function $-t_n(M(\cdot))$ is submodular, plus equality in some prescribed cases.

Furthermore, by looking at the 3-element minors of the matroid M , we get a set of relations, which are tropicalizations of Plücker relations, so that:

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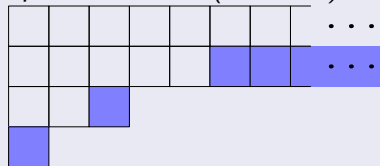
The vector $(t_n(M(A)), |A| = k)$ defines a point on the *Dressian** $Dr(k, |E|)$

In fact, we conjecture that in this way we get a point on the *Dressian analogue of the full flag variety**. (* tropical varieties parametrizing tropical linear spaces, and full flags of t.l.s., respectively).

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Valuated matroids

A **valuated matroid** is defined as a matroid decorated with an integer valued function \mathcal{V} on the set of the bases \mathcal{B} , satisfying a certain axiom [Dress and Wenzel]. There is a bijection

$$\{\text{tropical linear spaces}\} \longleftrightarrow \{\text{valuated matroids}\}$$

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Let R be a Dedekind domain. Given an R -module N , let $N_{\text{tors}} \subseteq N$ denote the submodule of its torsion elements, and N_{proj} denote the projective module N/N_{tors} .

There is a function \det associating to every R -module an element of $\text{Pic}(R)$, the **Picard group** of R .

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If M is a matroid over \mathbb{Z} , we define the two functions

$$\text{cork}(A) = \text{rk}(M(A)_{\text{proj}}) \quad \text{and} \quad m(A) \doteq |M(A)_{\text{tors}}|.$$

As a corollary of the previous theorem, we can prove that (E, cork, m) is a “quasi-arithmetic matroid”, a structure closely related to the arithmetic matroids introduced by [D’Adderio-M].

An **arithmetic matroid** is a pair (M, m) , where M is a (classical) matroid and $m : 2^E \rightarrow \mathbb{Z}_{>0}$ a *multiplicity* function, such that [complicated axioms]

Notice that matroids over \mathbb{Z} and (quasi-)arithmetic matroids are *not* truly equivalent, since the information contained in the former is **richer**, since there are many groups with the same cardinality.

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Definition of the Tutte-Grothendieck ring

Several invariants can be associated to a classical matroid; the universal deletion-contraction invariant is the well-known *Tutte polynomial*. We will now define and compute the universal deletion-contraction invariant of matroids over any Dedekind domain R .

Essentially following Brylawski, define the **Tutte-Grothendieck ring** of matroids over R , $K(R\text{-Mat})$, to be the abelian group generated by a symbol \mathbf{T}_M for each matroid M over R , modulo the relations

$$\mathbf{T}_M = \mathbf{T}_{M \setminus a} + \mathbf{T}_{M/a}$$

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When R is a field, $\text{Pic}(R)$ is trivial and there is no torsion, thus $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}] \simeq \mathbb{Z}[X, Y]$.

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The Tutte quasi-polynomial

Another invariant that we can obtain from the Grothendieck-Tutte invariant \mathbf{T}_M in the case $R = \mathbb{Z}$ is the **Tutte quasi-polynomial**

$$\mathbf{Q}_M(x, y) = \sum_{A \subseteq E} \frac{|M(A)_{\text{tors}}|}{|q \cdot M(A)_{\text{tors}}|} (x - 1)^{\text{rk}(E) - \text{rk}(A)} (y - 1)^{|A| - \text{rk}(A)},$$

where $q = (x - 1)(y - 1)$.

This is a quasi-polynomial in q , interpolating between the classical and the arithmetic Tutte polynomials.

This polynomial was introduced in [Brändén- M.], and has application to generalized colorings and flows on graphs with labeled edges.

Notice that $\mathbf{Q}_M(x, y)$ is not an invariant of the arithmetic matroid, (as it depends on the groups $M(A)_{\text{tors}}$ and not just on their cardinalities), but it is an invariant of the matroid over \mathbb{Z} .

The Tutte quasi-polynomial

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Developments and applications

Future developments:

- study other examples, such as R coordinate ring of an algebraic curve (e.g. the affine line or an elliptic curve);
- provide cryptomorphic definitions (e.g. the base polytope; a semi-standard Young tableaux description);

Possible applications:

- combinatorial topology: [Bajo-Burdick-Chmutov], [Duval-Klivans-Martin], [Hughes-Swartz], [Cavazzani- M.];
- tropical geometry;
- intersection theory for arrangements of subtori, toric varieties, ...;
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