

Linear rewriting and homology of associative algebras

Philippe Malbos

Institut Camille Jordan, Université Claude Bernard Lyon 1

Joint work with Yves Guiraud and Eric Hoffbeck

**MAP 2014 –
Mathematics Algorithms and Proofs**

May 30th, 2014, Paris

Motivation

- Let \mathbf{A} be an associative \mathbb{K} -algebra presented by generators V and relations R

$$\mathbf{A} = T(V)/\langle R \rangle.$$

Motivation

- ▶ Let \mathbf{A} be an associative \mathbb{K} -algebra presented by generators V and relations R

$$\mathbf{A} = T(V)/\langle R \rangle.$$

- ▶ Compute small projectives resolutions of \mathbb{K} as a \mathbf{A} -module.
 - ▷ Compute homology groups, Poincaré-Betti series, Koszul property...

Motivation

- ▶ Let \mathbf{A} be an associative \mathbb{K} -algebra presented by generators V and relations R

$$\mathbf{A} = T(V)/\langle R \rangle.$$

- ▶ Compute small projectives resolutions of \mathbb{K} as a \mathbf{A} -module.
 - ▷ Compute homology groups, Poincaré-Betti series, Koszul property...
- ▶ Methods based on Gröbner bases ([Anick](#), 1986, [Green](#), 1999, ...)

Motivation

- ▶ Let \mathbf{A} be an associative \mathbb{K} -algebra presented by generators V and relations R

$$\mathbf{A} = T(V)/\langle R \rangle.$$

- ▶ Compute small projectives resolutions of \mathbb{K} as a \mathbf{A} -module.
 - ▷ Compute homology groups, Poincaré-Betti series, Koszul property...
- ▶ Methods based on Gröbner bases ([Anick](#), 1986, [Green](#), 1999, ...)
- ▶ [Anick's](#) resolution: given a Gröbner basis G for R

Motivation

- ▶ Let \mathbf{A} be an associative \mathbb{K} -algebra presented by generators V and relations R

$$\mathbf{A} = T(V)/\langle R \rangle.$$

- ▶ Compute small projectives resolutions of \mathbb{K} as a \mathbf{A} -module.
 - ▷ Compute homology groups, Poincaré-Betti series, Koszul property...
- ▶ Methods based on Gröbner bases ([Anick](#), 1986, [Green](#), 1999, ...)
- ▶ [Anick's](#) resolution: given a Gröbner basis G for R

$$0 \longleftarrow \mathbb{K} \xleftarrow{\varepsilon} \mathbf{A} \xleftarrow{\delta_0} \mathbf{A} \otimes V \xleftarrow{\delta_1} \mathbf{A} \otimes F_1 \longleftarrow \dots \longleftarrow \mathbf{A} \otimes F_{n-1} \xleftarrow{\delta_n} \mathbf{A} \otimes F_n \longleftarrow \dots$$

where

- ▷ $F_n = \mathbb{K}\mathcal{C}_n$ is the free space on n -fold overlapping,
- ▷ the boundary map δ_n decomposes n -fold overlappings into $(n-1)$ -fold overlappings.

Motivation

- ▶ Let \mathbf{A} be an associative \mathbb{K} -algebra presented by generators V and relations R

$$\mathbf{A} = T(V)/\langle R \rangle.$$

- ▶ Compute small projectives resolutions of \mathbb{K} as a \mathbf{A} -module.
 - ▷ Compute homology groups, Poincaré-Betti series, Koszul property...
- ▶ Methods based on Gröbner bases ([Anick](#), 1986, [Green](#), 1999, ...)
- ▶ [Anick's](#) resolution: given a Gröbner basis G for R

$$0 \longleftarrow \mathbb{K} \xleftarrow{\varepsilon} \mathbf{A} \xleftarrow{\delta_0} \mathbf{A} \otimes V \xleftarrow{\delta_1} \mathbf{A} \otimes F_1 \longleftarrow \dots \longleftarrow \mathbf{A} \otimes F_{n-1} \xleftarrow{\delta_n} \mathbf{A} \otimes F_n \longleftarrow \dots$$

where

- ▶ $F_n = \mathbb{K}\mathcal{C}_n$ is the free space on **n -fold overlapping**,
 - ▶ the boundary map δ_n decomposes n -fold overlappings into $(n-1)$ -fold overlappings.
- ▶ Bergman software in Standard Lisp, <http://servus.math.su.se/bergman>, ([Backelin](#)).

Motivation

Example.

$$\mathbf{A} \langle x, y, z \mid xyz = x^3 + y^3 + z^3 \rangle$$

Motivation

Example.

$$\mathbf{A} \langle x, y, z \mid xyz = x^3 + y^3 + z^3 \rangle$$

► We have

$$\triangleright V = \mathbb{K} \langle x, y, z \rangle,$$

Motivation

Example.

$$\mathbf{A}\langle x, y, z \mid xyz = x^3 + y^3 + z^3 \rangle$$

► We have

▷ $V = \mathbb{K}\langle x, y, z \rangle,$

▷ Gröbner basis with lexicographic order with $x < y < z$

$$G = \left\{ z^3 - xyz + x^3 + y^3, zy^3 - zxyz + zx^3 + xyz^2 - x^3z - y^3z \right\}$$

Motivation

Example.

$$\mathbf{A}\langle x, y, z \mid xyz = x^3 + y^3 + z^3 \rangle$$

► We have

▷ $V = \mathbb{K}\langle x, y, z \rangle,$

▷ Gröbner basis with lexicographic order with $x < y < z$

$$G = \left\{ z^3 - xyz + x^3 + y^3, zy^3 - zxyz + zx^3 + xyz^2 - x^3z - y^3z \right\}$$

▷ $F_1 = \mathbb{K}\langle z^3, zy^3 \rangle,$

Motivation

Example.

$$\mathbf{A} \langle x, y, z \mid xyz = x^3 + y^3 + z^3 \rangle$$

► We have

▷ $V = \mathbb{K} \langle x, y, z \rangle,$

▷ Gröbner basis with lexicographic order with $x < y < z$

$$G = \left\{ z^3 - xyz + x^3 + y^3, zy^3 - zxyz + zx^3 + xyz^2 - x^3z - y^3z \right\}$$

▷ $F_1 = \mathbb{K} \langle z^3, zy^3 \rangle,$

▷ $F_2 = \mathbb{K} \langle z^4, z^3y^3 \rangle,$

Motivation

Example.

$$\mathbf{A}\langle x, y, z \mid xyz = x^3 + y^3 + z^3 \rangle$$

► We have

▷ $V = \mathbb{K}\langle x, y, z \rangle,$

▷ Gröbner basis with lexicographic order with $x < y < z$

$$G = \left\{ z^3 - xyz + x^3 + y^3, zy^3 - zxyz + zx^3 + xyz^2 - x^3z - y^3z \right\}$$

▷ $F_1 = \mathbb{K}\langle z^3, zy^3 \rangle,$

▷ $F_2 = \mathbb{K}\langle z^4, z^3y^3 \rangle,$

▷ $F_3 = \mathbb{K}\langle z^5, z^4y^3 \rangle,$

▷ ... giving an infinite free resolution.

Motivation

- ▶ Develop methods of **higher-dimensional rewriting** for algebras (and operads).

Motivation

- ▶ Develop methods of **higher-dimensional rewriting** for algebras (and operads).
- ▶ Computation of free resolutions for algebra presented by convergent presentations.
 - ▷ Does not require a monomial termination order.

Motivation

- ▶ Develop methods of **higher-dimensional rewriting** for algebras (and operads).
- ▶ Computation of free resolutions for algebra presented by convergent presentations.
 - ▷ Does not require a monomial termination order.
- ▶ Apply methods of higher-dimensional rewriting ([Guiraud-M.](#), 2012).

Motivation

- ▶ Develop methods of **higher-dimensional rewriting** for algebras (and operads).
- ▶ Computation of free resolutions for algebra presented by convergent presentations.
 - ▷ Does not require a monomial termination order.
- ▶ Apply methods of higher-dimensional rewriting ([Guiraud-M.](#), 2012).
- ▶ Application to Koszul property (computation of minimal free resolutions).

Motivation

- ▶ Develop methods of **higher-dimensional rewriting** for algebras (and operads).
- ▶ Computation of free resolutions for algebra presented by convergent presentations.
 - ▷ Does not require a monomial termination order.
- ▶ Apply methods of higher-dimensional rewriting ([Guiraud-M.](#), 2012).
- ▶ Application to Koszul property (computation of minimal free resolutions).

Example.

$$\mathbf{A} \langle x, y, z \mid xyz = x^3 + y^3 + z^3 \rangle$$

Motivation

- ▶ Develop methods of **higher-dimensional rewriting** for algebras (and operads).
- ▶ Computation of free resolutions for algebra presented by convergent presentations.
 - ▷ Does not require a monomial termination order.
- ▶ Apply methods of higher-dimensional rewriting ([Guiraud-M.](#), 2012).
- ▶ Application to Koszul property (computation of minimal free resolutions).

Example.

$$\mathbf{A} \langle x, y, z \mid xyz = x^3 + y^3 + z^3 \rangle$$

- ▶ A convergent presentation of **A**:

$$xyz \implies x^3 + y^3 + z^3.$$

Motivation

- ▶ Develop methods of **higher-dimensional rewriting** for algebras (and operads).
- ▶ Computation of free resolutions for algebra presented by convergent presentations.
 - ▷ Does not require a monomial termination order.
- ▶ Apply methods of higher-dimensional rewriting ([Guiraud-M., 2012](#)).
- ▶ Application to Koszul property (computation of minimal free resolutions).

Example.

$$\mathbf{A} \langle x, y, z \mid xyz = x^3 + y^3 + z^3 \rangle$$

- ▶ A convergent presentation of \mathbf{A} :

$$xyz \implies x^3 + y^3 + z^3.$$

- ▷ It is trivially convergent (no critical branching).
- ▷ We will deduce that the algebra \mathbf{A} is of cohomological dimension 2 and Koszul.

Plan

I. Linear Rewriting Systems

II. Computation of polygraphic resolutions

III. Free resolutions of algebras

References.

- [Y. Guiraud](#), [Eric Hoffbeck](#), [P.M.](#), Linear polygraphs and Koszulity of algebras, 2014, arXiv:1406.0815.
- [Y. Guiraud](#), [P.M.](#), Higher-dimensional normalisation strategies for acyclicity, 2012, arXiv:1011.0558.
- [Y. Guiraud](#), [P.M.](#) Polygraphs of finite derivation type, 2014, arXiv:1402.2587.

Part I. Linear rewriting systems

Rewriting

► **String rewriting systems** (Thue, 1914).

- ▷ Rewriting takes its origins in combinatorial algebra.
- ▷ Solving word problems, computing normal forms, ...

Rewriting

- ▶ **String rewriting systems** (Thue, 1914).
 - ▷ Rewriting takes its origins in combinatorial algebra.
 - ▷ Solving word problems, computing normal forms, ...

- ▶ **Abstract rewriting theory** (Newman, 1943).
 - ▷ Describe the properties of termination and of confluence.
 - ▷ Local to global theorem (Newman's diamond Lemma).

Rewriting

- ▶ **String rewriting systems** (Thue, 1914).
 - ▷ Rewriting takes its origins in combinatorial algebra.
 - ▷ Solving word problems, computing normal forms, ...

- ▶ **Abstract rewriting theory** (Newman, 1943).
 - ▷ Describe the properties of termination and of confluence.
 - ▷ Local to global theorem (Newman's diamond Lemma).

- ▶ **Gröbner bases** (Buchberger, 1965).
 - ▷ An algorithm for polynomial ideals.
 - ▷ For non commutative algebras (Shirshov, 1962, Bokut 1976, Bergman 1978, ...)

Rewriting

- ▶ **String rewriting systems** (Thue, 1914).
 - ▷ Rewriting takes its origins in combinatorial algebra.
 - ▷ Solving word problems, computing normal forms, ...

- ▶ **Abstract rewriting theory** (Newman, 1943).
 - ▷ Describe the properties of termination and of confluence.
 - ▷ Local to global theorem (Newman's diamond Lemma).

- ▶ **Gröbner bases** (Buchberger, 1965).
 - ▷ An algorithm for polynomial ideals.
 - ▷ For non commutative algebras (Shirshov, 1962, Bokut 1976, Bergman 1978, ...)

- ▶ **Polygraphs** (Burroni, 1991)
 - ▷ Higher-dimensional rewriting systems.

Rewriting

- ▶ **String rewriting systems** (Thue, 1914).
 - ▷ Rewriting takes its origins in combinatorial algebra.
 - ▷ Solving word problems, computing normal forms, ...

- ▶ **Abstract rewriting theory** (Newman, 1943).
 - ▷ Describe the properties of termination and of confluence.
 - ▷ Local to global theorem (Newman's diamond Lemma).

- ▶ **Gröbner bases** (Buchberger, 1965).
 - ▷ An algorithm for polynomial ideals.
 - ▷ For non commutative algebras (Shirshov, 1962, Bokut 1976, Bergman 1978, ...)

- ▶ **Polygraphs** (Burroni, 1991)
 - ▷ Higher-dimensional rewriting systems.
 - ▷ Geometrical structures of computation:
 - String rewriting systems, presentations of monoids or categories,
 - Term rewriting, presentations of Lawvere Theories and monoidal categories,
 - Automata, Petri nets, Turing Machine, ...

Linear rewriting systems

- Consider an oriented graph

$$\Sigma_0 \begin{array}{c} \xleftarrow{s_0} \\ \xleftarrow{t_0} \end{array} \Sigma_1$$

Linear rewriting systems

- ▶ Consider an oriented graph

$$\Sigma_0 \begin{array}{c} \xleftarrow{s_0} \\ \xleftarrow{t_0} \end{array} \Sigma_1$$

- ▶ Denote by Σ_1^ℓ the free algebroid on the graph (Σ_0, Σ_1) :

$$\Sigma_1^\ell(p, q) = \mathbb{K}\Sigma_1^*(p, q).$$

Linear rewriting systems

- ▶ Consider an oriented graph

$$\Sigma_0 \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{t_0} \end{array} \Sigma_1$$

- ▶ Denote by Σ_1^ℓ the free algebroid on the graph (Σ_0, Σ_1) :

$$\Sigma_1^\ell(p, q) = \mathbb{K}\Sigma_1^*(p, q).$$

- ▶ A (monic) **rule** is a 2-cell of the form

$$\begin{array}{ccc} & m & \\ q & \begin{array}{c} \curvearrowright \\ \Downarrow \alpha \\ \curvearrowleft \end{array} & q' \\ & h & \end{array}$$

where

- ▶ m is a non-zero monomial 1-cell in Σ_1^* ,
- ▶ $h = \sum_i \lambda_i m_i$ is a polynomial 1-cell.

Linear rewriting systems

- Consider an oriented graph

$$\Sigma_0 \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{t_0} \end{array} \Sigma_1$$

- Denote by Σ_1^ℓ the free algebroid on the graph (Σ_0, Σ_1) :

$$\Sigma_1^\ell(p, q) = \mathbb{K}\Sigma_1^*(p, q).$$

- A (monic) **rule** is a 2-cell of the form

$$q \begin{array}{c} \xrightarrow{m} \\ \Downarrow \alpha \\ \xrightarrow{h} \end{array} q'$$

where

- m is a non-zero monomial 1-cell in Σ_1^* ,
- $h = \sum_i \lambda_i m_i$ is a polynomial 1-cell.

- A **rewriting step** is a 2-cell of the form

$$\lambda \left(p \xrightarrow{m_1} q \begin{array}{c} \xrightarrow{m} \\ \Downarrow \alpha \\ \xrightarrow{h} \end{array} q' \xrightarrow{m_2} p' \right) + p \xrightarrow{g} p'$$

where

- λ is a non-zero scalar, m_1 and m_2 are non-zero monomials 1-cells in Σ_1^* ,
- $\alpha : m \Rightarrow h$ is a rule,

such that $m_1 m m_2$ does not appear in the basis decomposition of g .

Linear rewriting systems

- ▶ A **rewriting sequence** is a finite or an infinite sequence of rewriting steps

$$f_1 \Rightarrow f_2 \Rightarrow f_3 \Rightarrow \dots \Rightarrow f_n \Rightarrow \dots$$

Linear rewriting systems

- ▶ A **rewriting sequence** is a finite or an infinite sequence of rewriting steps

$$f_1 \Rightarrow f_2 \Rightarrow f_3 \Rightarrow \dots \Rightarrow f_n \Rightarrow \dots$$

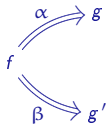
- ▶ The linear rewriting system **terminates** when it has no infinite rewriting sequence.

Linear rewriting systems

- ▶ A **rewriting sequence** is a finite or an infinite sequence of rewriting steps

$$f_1 \Rightarrow f_2 \Rightarrow f_3 \Rightarrow \dots \Rightarrow f_n \Rightarrow \dots$$

- ▶ The linear rewriting system **terminates** when it has no infinite rewriting sequence.
- ▶ A **branching** is a pair of rewriting sequences with a common source:

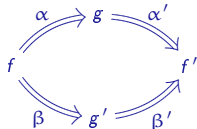


Linear rewriting systems

- ▶ A **rewriting sequence** is a finite or an infinite sequence of rewriting steps

$$f_1 \Rightarrow f_2 \Rightarrow f_3 \Rightarrow \dots \Rightarrow f_n \Rightarrow \dots$$

- ▶ The linear rewriting system **terminates** when it has no infinite rewriting sequence.
- ▶ A **branching** is a pair of rewriting sequences with a common source:



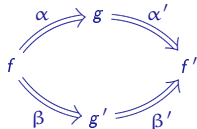
- ▶ It is **confluent** if there exists a pair (α', β') of rewriting sequences with a common target.

Linear rewriting systems

- ▶ A **rewriting sequence** is a finite or an infinite sequence of rewriting steps

$$f_1 \Rightarrow f_2 \Rightarrow f_3 \Rightarrow \dots \Rightarrow f_n \Rightarrow \dots$$

- ▶ The linear rewriting system **terminates** when it has no infinite rewriting sequence.
- ▶ A **branching** is a pair of rewriting sequences with a common source:



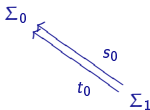
- ▶ It is **confluent** if there exists a pair (α', β') of rewriting sequences with a common target.
- ▶ The linear rewriting system is **convergent** when it terminates and all of its branchings are confluent.
 - ▶ In this case, any 1-cell u has a unique **normal form** \hat{u} .

Linear 2-polygraphs

- ▶ A linear rewriting system forms a **linear 2-polygraph**, that is a data $(\Sigma_0, \Sigma_1, \Lambda_2)$ where

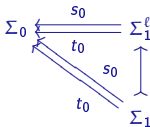
Linear 2-polygraphs

- ▶ A linear rewriting system forms a **linear 2-polygraph**, that is a data $(\Sigma_0, \Sigma_1, \Lambda_2)$ where
 - ▷ (Σ_0, Σ_1) is a directed graph,



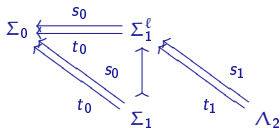
Linear 2-polygraphs

- ▶ A linear rewriting system forms a **linear 2-polygraph**, that is a data $(\Sigma_0, \Sigma_1, \Lambda_2)$ where
 - ▷ (Σ_0, Σ_1) is a directed graph,



Linear 2-polygraphs

- ▶ A linear rewriting system forms a **linear 2-polygraph**, that is a data $(\Sigma_0, \Sigma_1, \Lambda_2)$ where
 - ▷ (Σ_0, Σ_1) is a directed graph,
 - ▷ Λ_2 is a **linear cellular extension** of the free algebroid Σ_1^ℓ , that is a family of spaces $(\Lambda_2(p, q))_{p, q \in \Sigma_0}$ made of 2-cells generated by the set of rules Σ_2 .

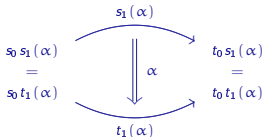
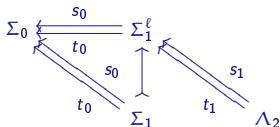


Linear 2-polygraphs

► A linear rewriting system forms a **linear 2-polygraph**, that is a data $(\Sigma_0, \Sigma_1, \Lambda_2)$ where

▷ (Σ_0, Σ_1) is a directed graph,

▷ Λ_2 is a **linear cellular extension** of the free algebroid Σ_1^ℓ , that is a family of spaces $(\Lambda_2(p, q))_{p, q \in \Sigma_0}$ made of 2-cells generated by the set of rules Σ_2 .



The 2-category of reductions

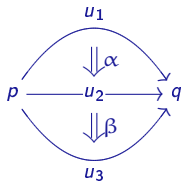
- ▶ Several compositions of rewriting 2-cells:

The 2-category of reductions

► Several compositions of rewriting 2-cells:

▷ the **sequential composition** $\alpha \star_1 \beta$ of rewriting steps is associative and unitary

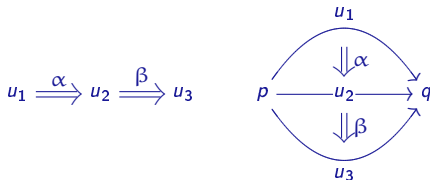
$$u_1 \xRightarrow{\alpha} u_2 \xRightarrow{\beta} u_3$$



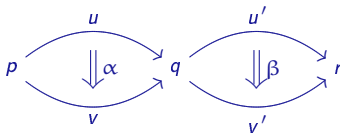
The 2-category of reductions

► Several compositions of rewriting 2-cells:

▷ the **sequential composition** $\alpha \star_1 \beta$ of rewriting steps is associative and unitary



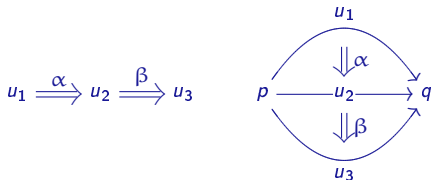
▷ the **parallel composition** $\alpha \star_0 \beta$ of rewriting steps is associative and unitary



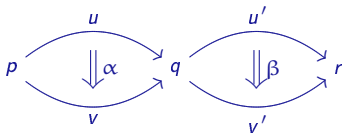
The 2-category of reductions

► Several compositions of rewriting 2-cells:

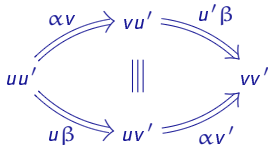
▷ the **sequential composition** $\alpha \star_1 \beta$ of rewriting steps is associative and unitary



▷ the **parallel composition** $\alpha \star_0 \beta$ of rewriting steps is associative and unitary



and compatible with the sequential composition (**exchange relations**)



$$\alpha v \star_1 u' \beta = u \beta \star_1 \alpha v'$$

The 2-algebroid of rewriting

► The linear structure on 1-cells induces a linear structure on 2-cells:

$$\lambda \left(\begin{array}{ccc} & u & \\ & \curvearrowright & \\ p & & q \\ & \Downarrow \alpha & \\ & \curvearrowleft & \\ & v & \end{array} \right) + \mu \left(\begin{array}{ccc} & u' & \\ & \curvearrowright & \\ p & & q \\ & \Downarrow \beta & \\ & \curvearrowleft & \\ & v' & \end{array} \right) = \begin{array}{ccc} & \lambda u + \mu u' & \\ & \curvearrowright & \\ p & & q \\ & \Downarrow \lambda \alpha + \mu \beta & \\ & \curvearrowleft & \\ & \lambda v + \mu v' & \end{array}$$

The 2-algebra of rewriting

► The linear structure on 1-cells induces a linear structure on 2-cells:

$$\lambda \left(\begin{array}{ccc} & u & \\ & \curvearrowright & \\ p & & q \\ & \Downarrow \alpha & \\ & \curvearrowleft & \\ & v & \end{array} \right) + \mu \left(\begin{array}{ccc} & u' & \\ & \curvearrowright & \\ p & & q \\ & \Downarrow \beta & \\ & \curvearrowleft & \\ & v' & \end{array} \right) = \begin{array}{ccc} & \lambda u + \mu u' & \\ & \curvearrowright & \\ p & & q \\ & \Downarrow \lambda \alpha + \mu \beta & \\ & \curvearrowleft & \\ & \lambda v + \mu v' & \end{array}$$

► compatible with composition maps \star_0 and \star_1 :

$$(\lambda \alpha + \mu \beta) \star_0 (\lambda' \alpha' + \mu' \beta') = \lambda \lambda' (\alpha \star_0 \alpha') + \lambda \mu' (\alpha \star_0 \beta') + \mu \lambda' (\alpha \star_0 \beta') + \mu \mu' (\beta \star_0 \beta')$$

$$(\lambda \alpha + \mu \beta) \star_1 (\lambda \alpha' + \mu \beta') = \lambda (\alpha \star_1 \alpha') + \mu (\beta \star_1 \beta')$$

The 2-algebroid of rewriting

► The linear structure on 1-cells induces a linear structure on 2-cells:

$$\lambda \left(\begin{array}{ccc} & u & \\ & \curvearrowright & \\ p & & q \\ & \Downarrow \alpha & \\ & \curvearrowleft & \\ & v & \end{array} \right) + \mu \left(\begin{array}{ccc} & u' & \\ & \curvearrowright & \\ p & & q \\ & \Downarrow \beta & \\ & \curvearrowleft & \\ & v' & \end{array} \right) = \begin{array}{ccc} & \lambda u + \mu u' & \\ & \curvearrowright & \\ p & & q \\ & \Downarrow \lambda \alpha + \mu \beta & \\ & \curvearrowleft & \\ & \lambda v + \mu v' & \end{array}$$

► compatible with composition maps \star_0 and \star_1 :

$$(\lambda \alpha + \mu \beta) \star_0 (\lambda' \alpha' + \mu' \beta') = \lambda \lambda' (\alpha \star_0 \alpha') + \lambda \mu' (\alpha \star_0 \beta') + \mu \lambda' (\alpha \star_0 \beta') + \mu \mu' (\beta \star_0 \beta')$$

$$(\lambda \alpha + \mu \beta) \star_1 (\lambda \alpha' + \mu \beta') = \lambda (\alpha \star_1 \alpha') + \mu (\beta \star_1 \beta')$$

► This forms a **2-algebroid**.

The 2-algebroid of rewriting

- The linear structure on 1-cells induces a linear structure on 2-cells:

$$\lambda \left(\begin{array}{ccc} & u & \\ & \curvearrowright & \\ p & & q \\ & \Downarrow \alpha & \\ & \curvearrowleft & \\ & v & \end{array} \right) + \mu \left(\begin{array}{ccc} & u' & \\ & \curvearrowright & \\ p & & q \\ & \Downarrow \beta & \\ & \curvearrowleft & \\ & v' & \end{array} \right) = \begin{array}{ccc} & \lambda u + \mu u' & \\ & \curvearrowright & \\ p & & q \\ & \Downarrow \lambda \alpha + \mu \beta & \\ & \curvearrowleft & \\ & \lambda v + \mu v' & \end{array}$$

- compatible with composition maps \star_0 and \star_1 :

$$(\lambda \alpha + \mu \beta) \star_0 (\lambda' \alpha' + \mu' \beta') = \lambda \lambda' (\alpha \star_0 \alpha') + \lambda \mu' (\alpha \star_0 \beta') + \mu \lambda' (\alpha \star_0 \beta') + \mu \mu' (\beta \star_0 \beta')$$

$$(\lambda \alpha + \mu \beta) \star_1 (\lambda \alpha' + \mu \beta') = \lambda (\alpha \star_1 \alpha') + \mu (\beta \star_1 \beta')$$

- This forms a **2-algebroid**.
- Will denote by Λ_2^ℓ the free 2-algebroid generated by the linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$.

The 2-algebra of rewriting

- The linear structure on 1-cells induces a linear structure on 2-cells:

$$\lambda \left(p \begin{array}{c} \xrightarrow{u} \\ \Downarrow \alpha \\ \xrightarrow{v} \end{array} q \right) + \mu \left(p \begin{array}{c} \xrightarrow{u'} \\ \Downarrow \beta \\ \xrightarrow{v'} \end{array} q \right) = p \begin{array}{c} \xrightarrow{\lambda u + \mu u'} \\ \Downarrow \lambda \alpha + \mu \beta' \\ \xrightarrow{\lambda v + \mu v'} \end{array} q$$

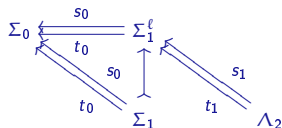
- compatible with composition maps \star_0 and \star_1 :

$$(\lambda \alpha + \mu \beta) \star_0 (\lambda' \alpha' + \mu' \beta') = \lambda \lambda' (\alpha \star_0 \alpha') + \lambda \mu' (\alpha \star_0 \beta') + \mu \lambda' (\alpha \star_0 \beta') + \mu \mu' (\beta \star_0 \beta')$$

$$(\lambda \alpha + \mu \beta) \star_1 (\lambda \alpha' + \mu \beta') = \lambda (\alpha \star_1 \alpha') + \mu (\beta \star_1 \beta')$$

- This forms a **2-algebra**.

- Will denote by Λ_2^ℓ the free 2-algebra generated by the linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$.



The 2-algebra of rewriting

- The linear structure on 1-cells induces a linear structure on 2-cells:

$$\lambda \left(p \begin{array}{c} \xrightarrow{u} \\ \Downarrow \alpha \\ \xrightarrow{v} \end{array} q \right) + \mu \left(p \begin{array}{c} \xrightarrow{u'} \\ \Downarrow \beta \\ \xrightarrow{v'} \end{array} q \right) = p \begin{array}{c} \xrightarrow{\lambda u + \mu u'} \\ \Downarrow \lambda \alpha + \mu \beta' \\ \xrightarrow{\lambda v + \mu v'} \end{array} q$$

- compatible with composition maps \star_0 and \star_1 :

$$(\lambda \alpha + \mu \beta) \star_0 (\lambda' \alpha' + \mu' \beta') = \lambda \lambda' (\alpha \star_0 \alpha') + \lambda \mu' (\alpha \star_0 \beta') + \mu \lambda' (\alpha \star_0 \beta') + \mu \mu' (\beta \star_0 \beta')$$

$$(\lambda \alpha + \mu \beta) \star_1 (\lambda \alpha' + \mu \beta') = \lambda (\alpha \star_1 \alpha') + \mu (\beta \star_1 \beta')$$

- This forms a **2-algebra**.

- Will denote by Λ_2^ℓ the free 2-algebra generated by the linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$.

$$\begin{array}{ccccc} \Sigma_0 & \xleftarrow{s_0} & \Sigma_1^\ell & \xleftarrow{s_1} & \Lambda_2^\ell \\ & \searrow t_0 & \uparrow & \searrow t_1 & \uparrow \\ & & \Sigma_1 & & \Lambda_2 \end{array}$$

Part II. Computation of polygraphic resolutions

Polygraphic resolutions

Aim. Starting with a presentation of an algebra \mathbf{A} by generators and relations (a linear 2-polygraph), we would like to compute a small categorical globular model for the algebra.

Polygraphic resolutions

Aim. Starting with a presentation of an algebra A by generators and relations (a linear 2-polygraph), we would like to compute a small categorical globular model for the algebra.

Method.

- ▶ Complete the presentation in a convergent presentation (Knuth-Bendix completion).

Polygraphic resolutions

Aim. Starting with a presentation of an algebra A by generators and relations (a linear 2-polygraph), we would like to compute a small categorical globular model for the algebra.

Method.

- ▷ Complete the presentation in a convergent presentation (Knuth-Bendix completion).
- ▷ Extend this presentation into an acyclic higher-dimensional polygraph
 - describing relations among relations, ...

Polygraphic resolutions

► A **polygraphic resolution** of length n of an algebra \mathbf{A} is a linear n -polygraph

$$(\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$$

Polygraphic resolutions

- A **polygraphic resolution** of length n of an algebra \mathbf{A} is a linear n -polygraph

$$(\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$$

such that

- ▷ the linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of $\mathbf{A} = \Sigma_1^\ell / \Lambda_2$,
- ▷ it is **acyclic**.

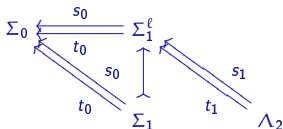
Polygraphic resolutions

► A **polygraphic resolution** of length n of an algebra \mathbf{A} is a linear n -polygraph

$$(\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$$

such that

- the linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of $\mathbf{A} = \Sigma_1^\ell / \Lambda_2$,
- it is **acyclic**.



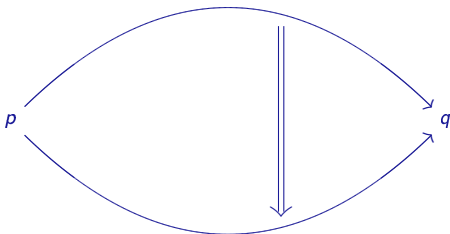
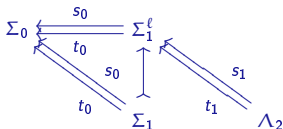
Polygraphic resolutions

► A **polygraphic resolution** of length n of an algebra \mathbf{A} is a linear n -polygraph

$$(\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$$

such that

- the linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of $\mathbf{A} = \Sigma_1^\ell / \Lambda_2$,
- it is **acyclic**.



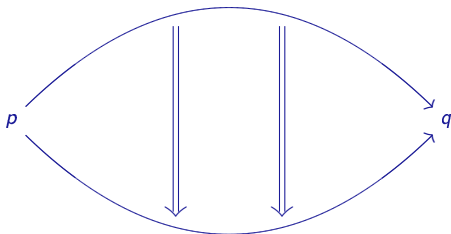
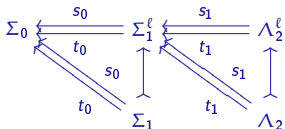
Polygraphic resolutions

► A **polygraphic resolution** of length n of an algebra \mathbf{A} is a linear n -polygraph

$$(\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$$

such that

- the linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of $\mathbf{A} = \Sigma_1^\ell / \Lambda_2$,
- it is **acyclic**.



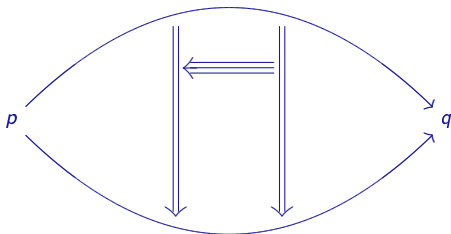
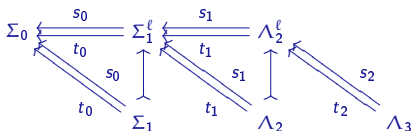
Polygraphic resolutions

► A **polygraphic resolution** of length n of an algebra \mathbf{A} is a linear n -polygraph

$$(\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$$

such that

- the linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of $\mathbf{A} = \Sigma_1^\ell / \Lambda_2$,
- it is **acyclic**.



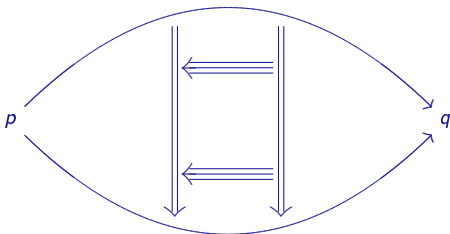
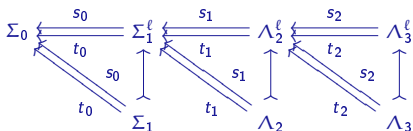
Polygraphic resolutions

► A **polygraphic resolution** of length n of an algebra \mathbf{A} is a linear n -polygraph

$$(\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$$

such that

- the linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of $\mathbf{A} = \Sigma_1^\ell / \Lambda_2$,
- it is **acyclic**.



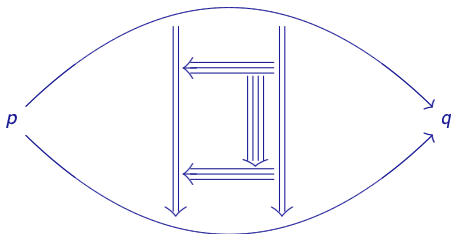
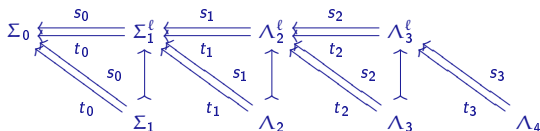
Polygraphic resolutions

► A **polygraphic resolution** of length n of an algebra \mathbf{A} is a linear n -polygraph

$$(\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$$

such that

- ▷ the linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of $\mathbf{A} = \Sigma_1^\ell / \Lambda_2$,
- ▷ it is **acyclic**.



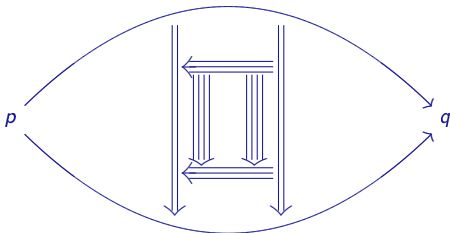
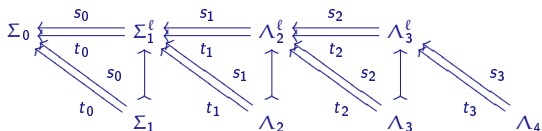
Polygraphic resolutions

► A **polygraphic resolution** of length n of an algebra \mathbf{A} is a linear n -polygraph

$$(\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$$

such that

- the linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of $\mathbf{A} = \Sigma_1^\ell / \Lambda_2$,
- it is **acyclic**.



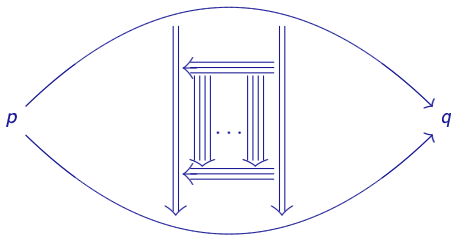
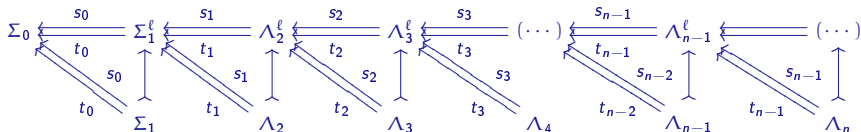
Polygraphic resolutions

► A **polygraphic resolution** of length n of an algebra \mathbf{A} is a linear n -polygraph

$$(\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$$

such that

- ▷ the linear 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of $\mathbf{A} = \Sigma_1^\ell / \Lambda_2$,
- ▷ it is **acyclic**.



Normalisation strategies

- ▶ **Normalisation strategies** give a constructive method to compute polygraphic resolutions.

Normalisation strategies

- ▶ **Normalisation strategies** give a constructive method to compute polygraphic resolutions.
- ▶ Let A be an algebra.

Normalisation strategies

- ▶ **Normalisation strategies** give a constructive method to compute polygraphic resolutions.
- ▶ Let \mathbf{A} be an algebra.
- ▶ Let $\Lambda = (\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$ be a linear n -polygraph,
 - ▶ such that the underlying 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of \mathbf{A} .

Normalisation strategies

- ▶ **Normalisation strategies** give a constructive method to compute polygraphic resolutions.
- ▶ Let \mathbf{A} be an algebra.
- ▶ Let $\Lambda = (\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$ be a linear n -polygraph,
 - ▶ such that the underlying 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of \mathbf{A} .
- ▶ We fix a **section** of Λ , that is a choice of a representative 1-cell

$$p \xrightarrow{\hat{u}} q \quad \text{in } \Sigma_1^\ell,$$

for every 1-cell $p \xrightarrow{u} q$ in \mathbf{A} .

Normalisation strategies

► **Normalisation strategies** give a constructive method to compute polygraphic resolutions.

► Let \mathbf{A} be an algebra.

► Let $\Lambda = (\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$ be a linear n -polygraph,

▷ such that the underlying 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of \mathbf{A} .

► We fix a **section** of Λ , that is a choice of a representative 1-cell

$$p \xrightarrow{\hat{u}} q \quad \text{in } \Sigma_1^\ell,$$

for every 1-cell $p \xrightarrow{u} q$ in \mathbf{A} .

► Normalisation strategy

$$u \qquad v$$

Normalisation strategies

► **Normalisation strategies** give a constructive method to compute polygraphic resolutions.

► Let \mathbf{A} be an algebra.

► Let $\Lambda = (\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$ be a linear n -polygraph,

▷ such that the underlying 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of \mathbf{A} .

► We fix a **section** of Λ , that is a choice of a representative 1-cell

$$p \xrightarrow{\hat{u}} q \quad \text{in } \Sigma_1^\ell,$$

for every 1-cell $p \xrightarrow{u} q$ in \mathbf{A} .

► Normalisation strategy

$$u \qquad v$$

Normalisation strategies

► **Normalisation strategies** give a constructive method to compute polygraphic resolutions.

► Let \mathbf{A} be an algebra.

► Let $\Lambda = (\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$ be a linear n -polygraph,

▷ such that the underlying 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of \mathbf{A} .

► We fix a **section** of Λ , that is a choice of a representative 1-cell

$$p \xrightarrow{\hat{u}} q \quad \text{in } \Sigma_1^{\ell},$$

for every 1-cell $p \xrightarrow{u} q$ in \mathbf{A} .

► Normalisation strategy

$$u \xrightarrow{f} v$$

Normalisation strategies

- ▶ **Normalisation strategies** give a constructive method to compute polygraphic resolutions.
- ▶ Let \mathbf{A} be an algebra.
- ▶ Let $\Lambda = (\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$ be a linear n -polygraph,
 - ▶ such that the underlying 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of \mathbf{A} .
- ▶ We fix a **section** of Λ , that is a choice of a representative 1-cell

$$p \xrightarrow{\hat{u}} q \quad \text{in } \Sigma_1^\ell,$$

for every 1-cell $p \xrightarrow{u} q$ in \mathbf{A} .

- ▶ Normalisation strategy

$$u \xrightarrow{f} v$$

$$\hat{u} = \hat{v}$$

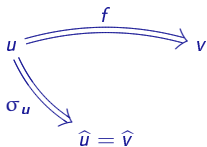
Normalisation strategies

- ▶ **Normalisation strategies** give a constructive method to compute polygraphic resolutions.
- ▶ Let \mathbf{A} be an algebra.
- ▶ Let $\Lambda = (\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$ be a linear n -polygraph,
 - ▶ such that the underlying 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of \mathbf{A} .
- ▶ We fix a **section** of Λ , that is a choice of a representative 1-cell

$$p \xrightarrow{\hat{u}} q \quad \text{in } \Sigma_1^\ell,$$

for every 1-cell $p \xrightarrow{u} q$ in \mathbf{A} .

- ▶ Normalisation strategy



Normalisation strategies

► **Normalisation strategies** give a constructive method to compute polygraphic resolutions.

► Let \mathbf{A} be an algebra.

► Let $\Lambda = (\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$ be a linear n -polygraph,

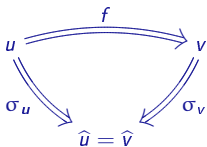
▷ such that the underlying 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of \mathbf{A} .

► We fix a **section** of Λ , that is a choice of a representative 1-cell

$$p \xrightarrow{\hat{u}} q \quad \text{in } \Sigma_1^\ell,$$

for every 1-cell $p \xrightarrow{u} q$ in \mathbf{A} .

► Normalisation strategy



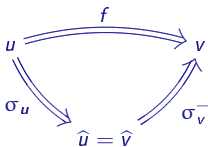
Normalisation strategies

- ▶ **Normalisation strategies** give a constructive method to compute polygraphic resolutions.
- ▶ Let \mathbf{A} be an algebra.
- ▶ Let $\Lambda = (\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$ be a linear n -polygraph,
 - ▶ such that the underlying 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of \mathbf{A} .
- ▶ We fix a **section** of Λ , that is a choice of a representative 1-cell

$$p \xrightarrow{\hat{u}} q \quad \text{in } \Sigma_1^\ell,$$

for every 1-cell $p \xrightarrow{u} q$ in \mathbf{A} .

- ▶ Normalisation strategy



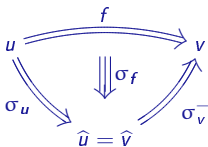
Normalisation strategies

- ▶ **Normalisation strategies** give a constructive method to compute polygraphic resolutions.
- ▶ Let \mathbf{A} be an algebra.
- ▶ Let $\Lambda = (\Sigma_0, \Sigma_1, \Lambda_2, \Lambda_3, \dots, \Lambda_n)$ be a linear n -polygraph,
 - ▶ such that the underlying 2-polygraph $(\Sigma_0, \Sigma_1, \Lambda_2)$ is a presentation of \mathbf{A} .
- ▶ We fix a **section** of Λ , that is a choice of a representative 1-cell

$$p \xrightarrow{\hat{u}} q \quad \text{in } \Sigma_1^\ell,$$

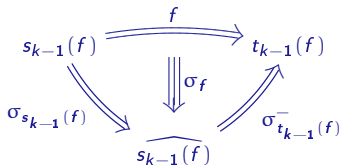
for every 1-cell $p \xrightarrow{u} q$ in \mathbf{A} .

- ▶ Normalisation strategy



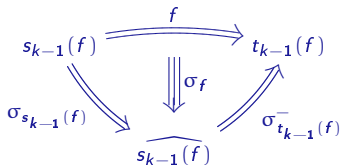
Normalisation strategies

- A **higher-dimensional normalisation strategy** for Λ is a mapping σ of every k -cell f to a $(k + 1)$ -cell:



Normalisation strategies

► A **higher-dimensional normalisation strategy** for Λ is a mapping σ of every k -cell f to a $(k + 1)$ -cell:

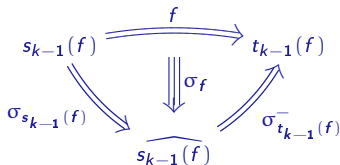


such that

$$\sigma_{\widehat{f}} = 1_{\widehat{f}}, \quad \sigma_{f+g} = \sigma_f + \sigma_g, \quad \sigma_{\lambda f} = \lambda \sigma_f.$$

Normalisation strategies

► A **higher-dimensional normalisation strategy** for Λ is a mapping σ of every k -cell f to a $(k + 1)$ -cell:



such that

$$\sigma_{\widehat{f}} = 1_{\widehat{f}}, \quad \sigma_{f+g} = \sigma_f + \sigma_g, \quad \sigma_{\lambda f} = \lambda \sigma_f.$$

Theorem. (G.-H.-M., 2014)

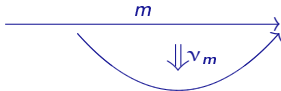
A linear n -polygraph is acyclic if, and only if, it admits a normalisation strategy.

The rightmost normalisation strategy

- ▶ Suppose that Λ is a reduced convergent linear 2-polygraph.

The rightmost normalisation strategy

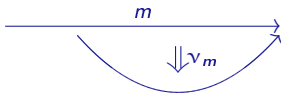
- ▶ Suppose that Λ is a reduced convergent linear 2-polygraph.
- ▶ Denote ν_m the **rightmost** rewriting step on a reducible monomial 1-cell m :



The rightmost normalisation strategy

► Suppose that \mathcal{A} is a reduced convergent linear 2-polygraph.

► Denote ν_m the **rightmost** rewriting step on a reducible monomial 1-cell m :

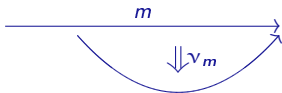


► Suppose that \mathcal{A} is equipped with its **rightmost normalisation strategy** ρ defined by

The rightmost normalisation strategy

► Suppose that \mathcal{A} is a reduced convergent linear 2-polygraph.

► Denote ν_m the **rightmost** rewriting step on a reducible monomial 1-cell m :



► Suppose that \mathcal{A} is equipped with its **rightmost normalisation strategy** ρ defined by

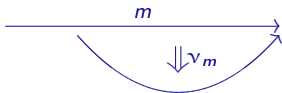
▷ on an irreducible monomial 1-cells m

$$\rho_m = 1_m$$

The rightmost normalisation strategy

► Suppose that \mathcal{A} is a reduced convergent linear 2-polygraph.

► Denote ν_m the **rightmost** rewriting step on a reducible monomial 1-cell m :



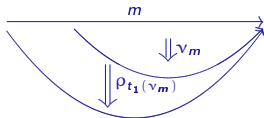
► Suppose that \mathcal{A} is equipped with its **rightmost normalisation strategy** ρ defined by

▷ on a irreducible monomial 1-cells m

$$\rho_m = 1_m$$

▷ on a reducible monomial 1-cells m

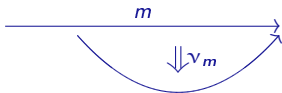
$$\rho_m = \nu_m * 1 \rho_{t_1(\nu_m)}$$



The rightmost normalisation strategy

► Suppose that \mathcal{A} is a reduced convergent linear 2-polygraph.

► Denote ν_m the **rightmost** rewriting step on a reducible monomial 1-cell m :



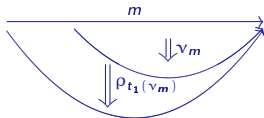
► Suppose that \mathcal{A} is equipped with its **rightmost normalisation strategy** ρ defined by

▷ on a irreducible monomial 1-cells m

$$\rho_m = 1_m$$

▷ on a reducible monomial 1-cells m

$$\rho_m = \nu_m * 1 \rho_{t_1(\nu_m)}$$

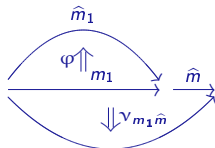


▷ on a 1-cell $f = \sum_i \lambda_i m_i$, by

$$\rho_f = \sum_i \lambda_i \rho_{m_i}.$$

Critical branchings

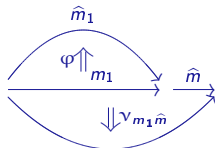
- Any **critical branching** has the following shape:



with φ in Λ_2 .

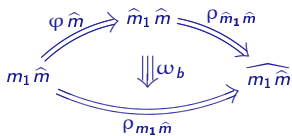
Critical branchings

- ▶ Any **critical branching** has the following shape:



with φ in Λ_2 .

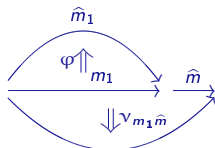
- ▶ We consider the linear cellular extension $\text{Gc}_3(\Lambda)$ of Λ_2^ℓ made of one 3-cell



for every critical branching b .

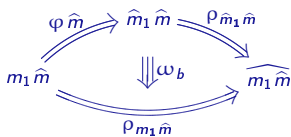
Critical branchings

- Any **critical branching** has the following shape:



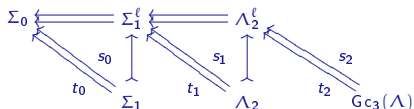
with φ in Λ_2 .

- We consider the linear cellular extension $G_{C_3}(\Lambda)$ of Λ_2^ℓ made of one 3-cell



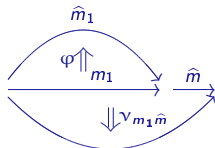
for every critical branching b .

Proposition. The following linear 3-polygraph is acyclic



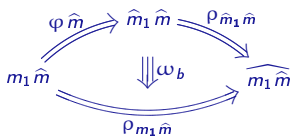
Critical branchings

- Any **critical branching** has the following shape:



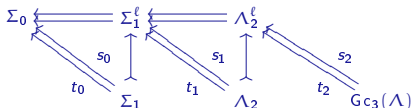
with φ in Λ_2 .

- We consider the linear cellular extension $G_{C_3}(\Lambda)$ of Λ_2^ℓ made of one 3-cell



for every critical branching b .

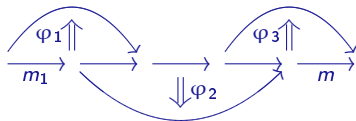
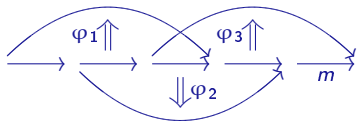
Proposition. The following linear 3-polygraph is acyclic



Proof (idea). We prove that the rightmost normalisation strategy ρ extends to a normalisation strategy of the linear 3-polygraph $G_{C_3}(\Lambda)$.

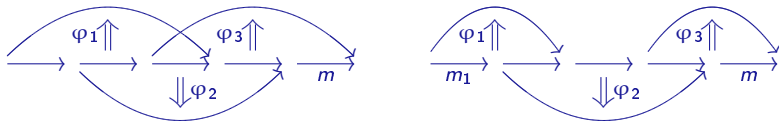
Higher-dimensional branchings

- A **3-fold branching** is an overlapping of three rewriting steps:



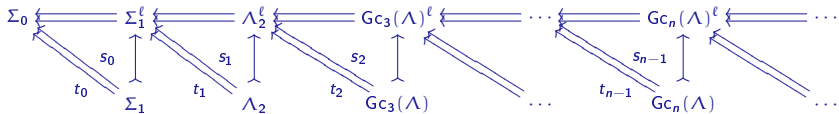
Higher-dimensional branchings

► A **3-fold branching** is an overlapping of three rewriting steps:



Theorem. (G.-H.-M., 2014)

Any convergent linear 2-polygraph Λ extends to an acyclic linear ∞ -polygraph $G_{C_\infty}(\Lambda)$



whose n -cells are indexed by the critical $(n - 1)$ -fold branchings.

Part III. Free resolutions of algebras

A free bimodules resolution

- ▶ Let A be an algebra.

A free bimodules resolution

- ▶ Let \mathbf{A} be an algebra.
- ▶ Let \mathcal{A} be a polygraphic resolution of \mathbf{A} .

A free bimodules resolution

- ▶ Let \mathbf{A} be an algebra.
- ▶ Let \mathcal{A} be a polygraphic resolution of \mathbf{A} .
- ▶ We denote by $\mathbf{A}^e[\mathcal{A}_k]$ the free \mathbf{A} -bimodule on \mathcal{A}_k .

A free bimodules resolution

- ▶ Let \mathbf{A} be an algebra.
- ▶ Let \mathcal{A} be a polygraphic resolution of \mathbf{A} .
- ▶ We denote by $\mathbf{A}^e[\mathcal{A}_k]$ the free \mathbf{A} -bimodule on \mathcal{A}_k .
- ▶ To the polygraphic resolution \mathcal{A} , we associate a complex of \mathbf{A} -bimodules

$$0 \longleftarrow \mathbf{A} \xleftarrow{\mu} \mathbf{A}^e[\mathcal{A}_0] \xleftarrow{\delta_0} \mathbf{A}^e[\mathcal{A}_1] \longleftarrow \dots \longleftarrow \mathbf{A}^e[\mathcal{A}_k] \xleftarrow{\delta_k} \mathbf{A}^e[\mathcal{A}_{k+1}] \longleftarrow \dots$$

- ▶ where the boundary maps are defined by

$$\mu(u \otimes v) = uv, \quad \delta_0[x] = 1 \otimes x - x \otimes 1, \quad \delta_k[f] = [s_k(f)] - [t_k(f)].$$

A free bimodules resolution

- ▶ Let \mathbf{A} be an algebra.
- ▶ Let \mathcal{A} be a polygraphic resolution of \mathbf{A} .
- ▶ We denote by $\mathbf{A}^e[\mathcal{A}_k]$ the free \mathbf{A} -bimodule on \mathcal{A}_k .
- ▶ To the polygraphic resolution \mathcal{A} , we associate a complex of \mathbf{A} -bimodules

$$0 \longleftarrow \mathbf{A} \xleftarrow{\mu} \mathbf{A}^e[\mathcal{A}_0] \xleftarrow{\delta_0} \mathbf{A}^e[\mathcal{A}_1] \longleftarrow \dots \longleftarrow \mathbf{A}^e[\mathcal{A}_k] \xleftarrow{\delta_k} \mathbf{A}^e[\mathcal{A}_{k+1}] \longleftarrow \dots$$

- ▶ where the boundary maps are defined by

$$\mu(u \otimes v) = uv, \quad \delta_0[x] = 1 \otimes x - x \otimes 1, \quad \delta_k[f] = [s_k(f)] - [t_k(f)].$$

- ▶ By the globular relations, we have $\delta^2 = 0$.

A free bimodules resolution

Proposition.

If \mathcal{A} is a polygraphic resolution of an algebra \mathbf{A} , then the complex $\mathbf{A}^e[\mathcal{A}]$ is a free resolution of the \mathbf{A} -bimodule \mathbf{A} .

A free bimodules resolution

Proposition.

If \mathcal{A} is a polygraphic resolution of an algebra \mathbf{A} , then the complex $\mathbf{A}^e[\mathcal{A}]$ is a free resolution of the \mathbf{A} -bimodule \mathbf{A} .

Proof (idea).

- ▷ Suppose that \mathcal{A} is a polygraphic resolution of the algebra \mathbf{A} .

A free bimodules resolution

Proposition.

If Λ is a polygraphic resolution of an algebra \mathbf{A} , then the complex $\mathbf{A}^e[\Lambda]$ is a free resolution of the \mathbf{A} -bimodule \mathbf{A} .

Proof (idea).

- ▷ Suppose that Λ is a polygraphic resolution of the algebra \mathbf{A} .
- ▷ The polygraph Λ being acyclic, it admits a normalisation strategy σ .

A free bimodules resolution

Proposition.

If Λ is a polygraphic resolution of an algebra \mathbf{A} , then the complex $\mathbf{A}^e[\Lambda]$ is a free resolution of the \mathbf{A} -bimodule \mathbf{A} .

Proof (idea).

- ▷ Suppose that Λ is a polygraphic resolution of the algebra \mathbf{A} .
- ▷ The polygraph Λ being acyclic, it admits a normalisation strategy σ .
- ▷ The strategy σ induces a contracting homotopy ι for the complex $\mathbf{A}^e[\Lambda]$

$$\iota_0 : \mathbf{A} \rightarrow \mathbf{A}^e[\Lambda_0], \quad \iota_k : \mathbf{A}^e[\Lambda_{k-1}] \rightarrow \mathbf{A}^e[\Lambda_1]$$

defined by

$$\iota_0(u) = u \otimes 1, \quad \iota_1(u \otimes v) = u[\widehat{v}], \quad \iota_k(u[f]v) = u[\sigma_{f\widehat{v}}].$$

Koszul algebras

Definition. A graded algebra \mathbf{A} is **Koszul** if

$$\mathrm{Tor}_{k,(i)}^{\mathbf{A}}(\mathbb{K}, \mathbb{K}) = 0, \quad \text{for } k \neq i.$$

Koszul algebras

Definition. A graded algebra A is **Koszul** if

$$\mathrm{Tor}_{k,(i)}^A(\mathbb{K}, \mathbb{K}) = 0, \quad \text{for } k \neq i.$$

Theorem. (Anick, 1986, Green 1999)

An algebra having a presentation by a quadratic Gröbner basis is Koszul.

Koszul algebras

Definition. A graded algebra A is **Koszul** if

$$\mathrm{Tor}_{k,(i)}^A(\mathbb{K}, \mathbb{K}) = 0, \quad \text{for } k \neq i.$$

Theorem. (Anick, 1986, Green 1999)

An algebra having a presentation by a quadratic Gröbner basis is Koszul.

Definition. (Berger 2001) A N -homogeneous algebra, with $N > 1$ is **Koszul** if

$$\mathrm{Tor}_{k,(i)}^A(\mathbb{K}, \mathbb{K}) = 0, \quad \text{for } i \neq \ell_N(k),$$

where

- ▷ k refers to the homological degree and (i) refers to the weight grading,
- ▷ ℓ_N is the weight function defined by:

$$\ell_N(k) = \begin{cases} IN & \text{if } k = 2l, \\ IN + 1 & \text{if } k = 2l + 1. \end{cases}$$

Convergence and Koszulity

Theorem. (G.-H.-M., 2014)

Let \mathbf{A} be a N -homogeneous algebra. If \mathbf{A} has a polygraphic resolution ℓ_N -concentrated, then it is Koszul.

Convergence and Koszulity

Theorem. (G.-H.-M., 2014)

Let \mathbf{A} be a N -homogeneous algebra. If \mathbf{A} has a polygraphic resolution ℓ_N -concentrated, then it is Koszul.

Corollary.

An algebra having a presentation by a quadratic convergent linear 2-polygraph is Koszul.

Convergence and Koszulity

Theorem. (G.-H.-M., 2014)

Let \mathbf{A} be a N -homogeneous algebra. If \mathbf{A} has a polygraphic resolution ℓ_N -concentrated, then it is Koszul.

Corollary.

An algebra having a presentation by a quadratic convergent linear 2-polygraph is Koszul.

Example.

The algebra $\mathbf{A}\langle x, y, z \mid xyz = x^3 + y^3 + z^3 \rangle$ is Koszul.

$$xyz \Rightarrow x^3 + y^3 + z^3$$

Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

- ▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

- ▷ The algebra \mathbf{A} is presented by the linear 2-polygraph

$$yz \xrightarrow{\alpha} -x^2 \quad zy \xrightarrow{\beta} -\frac{1}{a}x^2$$

Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph

$$yz \xrightarrow{\alpha} -x^2 \qquad zy \xrightarrow{\beta} -\frac{1}{a}x^2$$

$$\begin{array}{ccc} yzy & \xrightarrow{y\beta} & -\frac{1}{a}yx^2 \\ & \searrow \alpha y & \searrow \\ & & -x^2y \end{array} \qquad \begin{array}{ccc} zyz & \xrightarrow{\beta z} & -\frac{1}{a}x^2z \\ & \searrow z\alpha & \searrow \\ & & -zx^2 \end{array}$$

Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph

$$yz \xrightarrow{\alpha} -x^2 \qquad zy \xrightarrow{\beta} -\frac{1}{a}x^2 \qquad yx^2 \xrightarrow{\gamma} ax^2y \qquad zx^2 \xrightarrow{\delta} \frac{1}{a}x^2z$$

$$\begin{array}{ccc} yzy & \xrightarrow{y\beta} & -\frac{1}{a}yx^2 \\ & \searrow \alpha y & \downarrow -\frac{1}{a}\gamma \\ & & -x^2y \end{array} \qquad \begin{array}{ccc} zyz & \xrightarrow{\beta z} & -\frac{1}{a}x^2z \\ & \searrow z\alpha & \uparrow -\delta \\ & & -zx^2 \end{array}$$

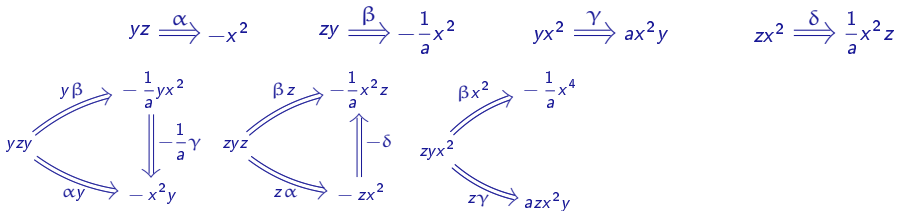
Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph



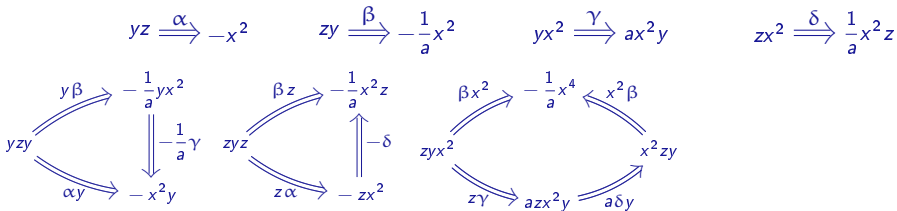
Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph



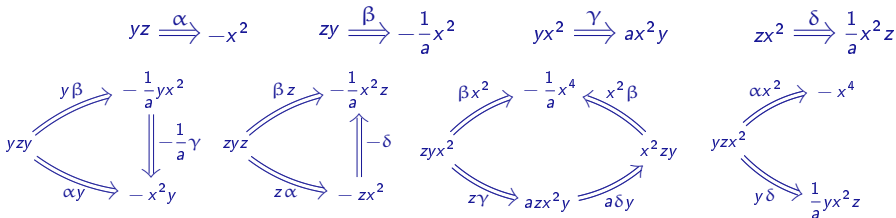
Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph



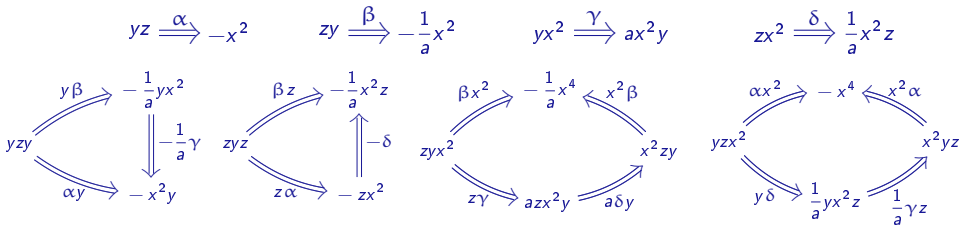
Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph



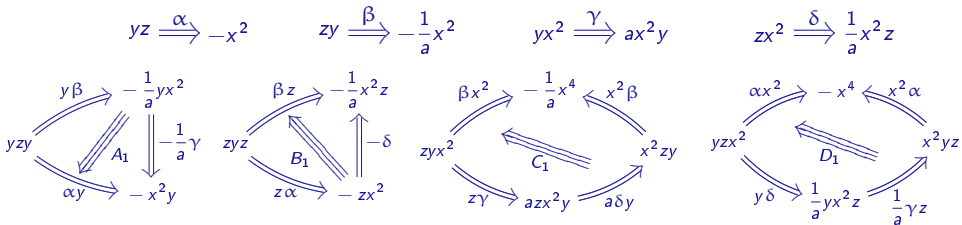
Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph



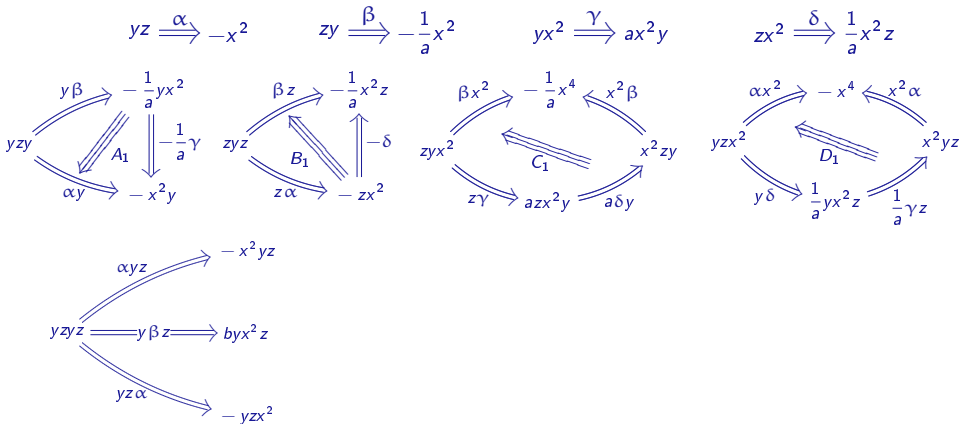
Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph



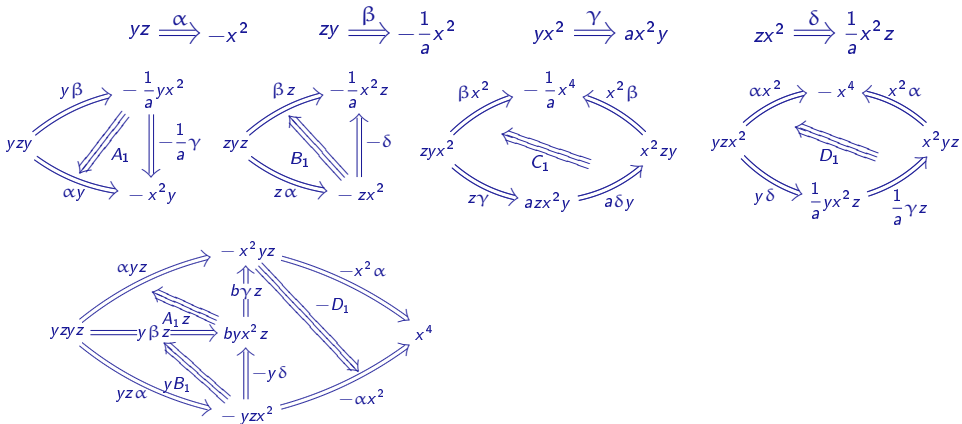
Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph



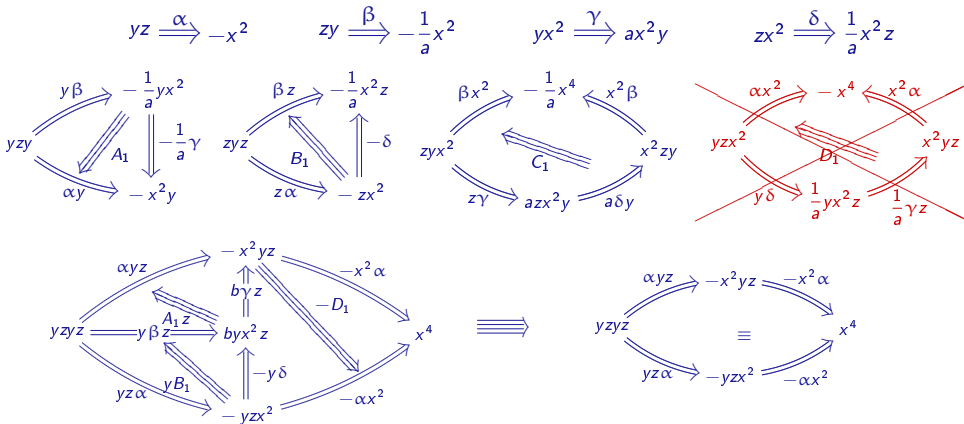
Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph



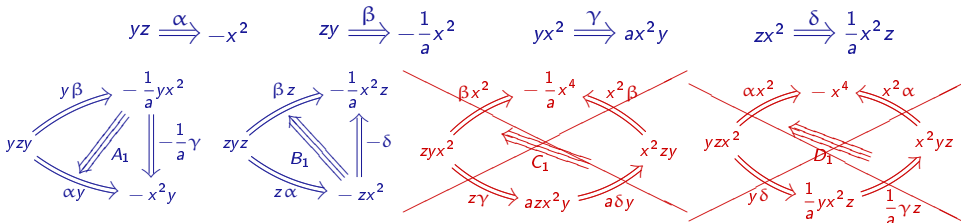
Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph



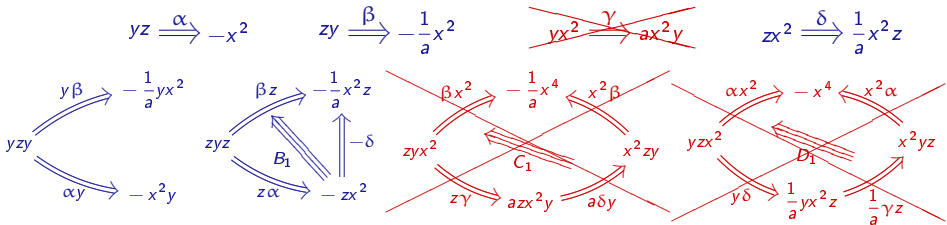
Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph



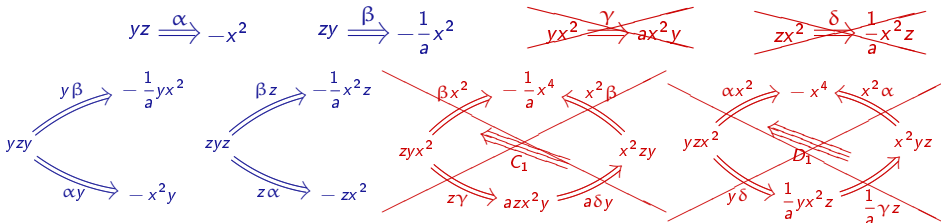
Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph



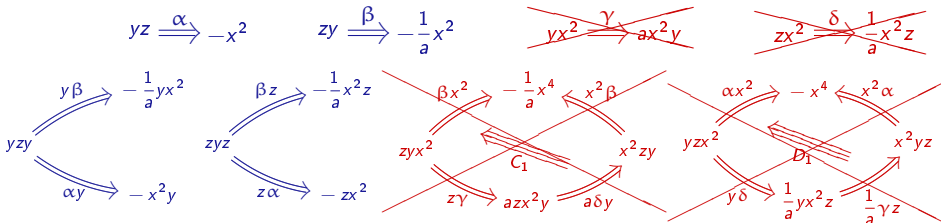
Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A} \langle x, y, z \mid x^2 + yz = 0, x^2 + azy = 0 \rangle \quad \text{with } a \neq 0, 1$$

▷ The algebra \mathbf{A} is presented by the linear 2-polygraph



▷ It follows that the algebra \mathbf{A} is Koszul.

▷ Note that

$$\mathrm{Tor}_{0,(0)}^{\mathbf{A}}(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K}, \quad \mathrm{Tor}_{1,(1)}^{\mathbf{A}}(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K}^3, \quad \mathrm{Tor}_{2,(2)}^{\mathbf{A}}(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K}^2,$$

$$\mathrm{Tor}_{k,(i)}^{\mathbf{A}}(\mathbb{K}, \mathbb{K}) = 0 \quad \text{otherwise.}$$