

**Theoretical study of steerable homogeneous operators**  
And applications to sparse stochastic processes  
Presentation – End of my Internship

Lilian Besson

Advisors: **Michael Unser** and **Julien Fageot**

**École polytechnique fédérale de Lausanne**  
**ENS de Cachan** (Master MVA)

August 26st, 2016 | Time : 40 minutes

E-mail : [lilian.besson@ens-cachan.fr](mailto:lilian.besson@ens-cachan.fr)  
Open-source : <http://lbo.k.vu/epfl2016>  
Grade: I got 20/20 for my internship

# Subject

## Functional operators

Mainly about *convolution* operators  $G$   
(= linear + continuous + translation-invariant, “LSI”)

## *Steerable* and *homogeneous* convolutions

- More freedom than if rotation-invariant
- But still easily parametrized !

# Subject

## Functional operators

Mainly about *convolution* operators  $G$   
(= linear + continuous + translation-invariant, “LSI”)

## *Steerable* and *homogeneous* convolutions

- More freedom than if rotation-invariant
- But still easily parametrized !

## Applications and experiments

Mainly on sparse stochastic processes, in 2D

# Apply our operators $G$ to ...

## Sparse processes

[Unser and Tafti, 2014]

- To visualize their effects
- To generate new processes
- ...and see pretty images ☺ !

# Apply our operators $G$ to ...

## Sparse processes

[Unser and Tafti, 2014]

- To visualize their effects
- To generate new processes
- ...and see pretty images ☺ !

## Splines

[Unser et al., 2016]

- One operator  $\iff$  one spline

# Outline

- 1 Reminders on operators theory
- 2 Steerable operators
- 3 Scale-invariance for steerable convolutions
- 4 Decompositions of steerable convolutions
- 5 Illustrations on sparse stochastic processes
- 6 Conclusion & Appendix

# ① Reminders on operators theory

# What are operators ?

## An operator $G$

Takes a function  $f$ , **transforms it** to another function  $G\{f\}$

For **real**-valued  $f$ ,  $G$  has to give a **real**-valued  $G\{f\}$



# What are operators ?

## An operator $G$

Takes a function  $f$ , **transforms it** to another function  $G\{f\}$   
For **real**-valued  $f$ ,  $G$  has to give a **real**-valued  $G\{f\}$

## Examples in maths

- Derivatives  $D_x$  and  $D_y$ , Laplacian  $\Delta$
- Denoising, contour detection, etc

## Examples in real life ?

- "Filters" for photos, in Instagram or Facebook !
- Rotating photos, etc

# Different properties for operators

Today, our operators  $G$  are *always*

- **Linear**
- **Continuous**
- **In 2D** :  $G : f(x, y) \mapsto G\{f\}(x, y)$
- **Translation-invariance**

# Different properties for operators

Today, our operators  $G$  are *always*

- **Linear**
- **Continuous**
- **In 2D** :  $G : f(x, y) \mapsto G\{f\}(x, y)$
- **Translation-invariance**

Geometric properties

[Unser and Tafti, 2014]

- **Scale-invariance** or  $\gamma$ -scale-invariance (= homogeneity)
- **Rotation-invariance**
- **Steerable**

# Schwartz theorem and impulse response

Schwartz convolution theorem

[Stein and Weiss, 1971]

Translation-invariant linear continuous operators are exactly  
**convolution operators** :

$$G\{f\} = (g * f)$$

# Schwartz theorem and impulse response

## Schwartz convolution theorem

[Stein and Weiss, 1971]

Translation-invariant linear continuous operators are exactly **convolution operators** :

$$G\{f\} = (g * f)$$

## Impulse response $g$ of $G$

$g = G\{\delta_0\}$  is a distribution (= generalized function)

# Fourier multiplier $\hat{g}$ of an operator $G$

Using the Fourier transform  $\mathcal{F}$  [Stein and Weiss, 1971]

$\mathcal{F}$  transforms a convolution  $(g * f)$  to a **point-wise product**

So

$$\mathcal{F}\{G\{f\}\} = \mathcal{F}\{g * f\} = \mathcal{F}\{g\} \cdot \mathcal{F}\{f\} = \hat{g} \cdot \hat{f}$$

Fourier multiplier  $\hat{g}$

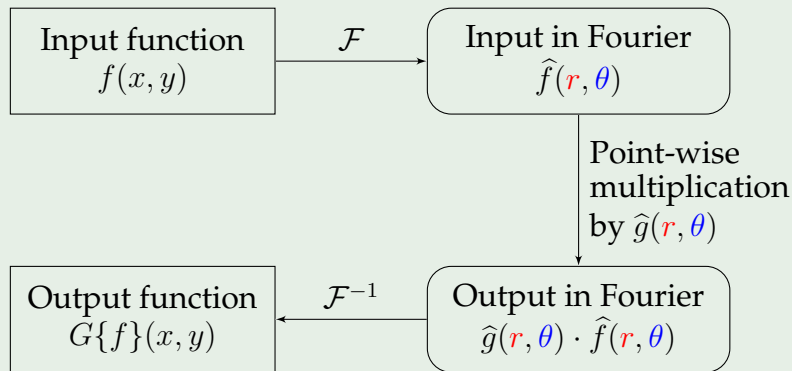
And so

$$G\{f\} = \mathcal{F}^{-1}\{\hat{g} \cdot \hat{f}\}$$

$\hat{g} = \mathcal{F}\{g\}$  is a **complex-valued** function

# Using the Fourier multiplier $\hat{g}$

## Working in the "Fourier domain"



In 2D, the Fourier variable  $\omega$  is written in polar coordinates :  $\omega = (r, \theta)$

# Property: scale-invariance

## Definition

For  $\gamma > 0$ ,  $G$  is  $\gamma$ -scale-invariant when

$$G\{f\}\left(\begin{bmatrix} x/a \\ y/a \end{bmatrix}\right) = a^\gamma G\{f(\cdot/a)\}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right), \quad \forall \text{ scaling } a > 0$$



# Property: scale-invariance

## Definition

For  $\gamma > 0$ ,  $G$  is  $\gamma$ -scale-invariant when

$$G\{f\}\left(\begin{bmatrix} x/a \\ y/a \end{bmatrix}\right) = a^\gamma G\{f(\cdot/a)\}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right), \quad \forall \text{ scaling } a > 0$$

## Easy characterization on $\hat{g}$

$$\hat{g}(ar, \theta) = a^\gamma \hat{g}(r, \theta), \quad \forall a > 0$$

## Separable form for $\hat{g}$

- $\hat{g}(r, \theta) = r^\gamma \hat{g}(1, \theta)$
- $\hat{g}(1, \theta)$  only depends on the angle  $\theta$

# Example #1: derivatives

## Derivatives and directional derivatives

- Usual derivatives :  $D_x$  and  $D_y$
- Directional derivative :  $D_\alpha \stackrel{\text{def}}{=} \cos(\alpha)D_x + \sin(\alpha)D_y$

## They are 1-scale-invariant

Because their Fourier multipliers are

- $\widehat{D_x}(r, \theta) = jx = jr \cos(\theta)$
- $\widehat{D_y}(r, \theta) = jy = jr \sin(\theta)$
- $\widehat{D_\alpha}(r, \theta) = jr \cos(\theta - \alpha)$

# Property: rotation-invariance

## Definition

$G$  is *rotation-invariant* when

$$G\{f\}(R_{\theta_0} \begin{bmatrix} x \\ y \end{bmatrix}) = G\{f(R_{\theta_0} \cdot)\}(\begin{bmatrix} x \\ y \end{bmatrix}) \quad \forall \text{ rotation } R_{\theta_0}$$

# Property: rotation-invariance

## Definition

$G$  is *rotation-invariant* when

$$G\{f\}(R_{\theta_0} \begin{bmatrix} x \\ y \end{bmatrix}) = G\{f(R_{\theta_0} \cdot)\}(\begin{bmatrix} x \\ y \end{bmatrix}) \quad \forall \text{ rotation } R_{\theta_0}$$

## Easy characterization on $\hat{g}$

$$\hat{g}(r, \theta + \theta_0) = \hat{g}(r, \theta), \quad \forall \text{ rotation angle } \theta_0$$

## Purely radial $\hat{g}$

$\hat{g}(r, \theta) = \hat{g}(r, 0)$  only depends on the radius  $r$  :  $\hat{g}$  is purely **radial** !

## Example #2: fractional Laplacians

Laplacian and fractional Laplacians [Unser and Tafti, 2014]

For  $\gamma > 0$ ,  $(-\Delta)^{\gamma/2}$  has a Fourier multiplier  $\widehat{g}(r, \theta) = r^\gamma$

So they are ...

- $\gamma$ -scale-invariant
  - and rotation-invariant
- $\implies$  Simplest example of  $\gamma$ -SI and RI operators !

In fact ...

## Example #2: fractional Laplacians

Laplacian and fractional Laplacians [Unser and Tafti, 2014]

For  $\gamma > 0$ ,  $(-\Delta)^{\gamma/2}$  has a Fourier multiplier  $\widehat{g}(r, \theta) = r^\gamma$

So they are ...

- $\gamma$ -scale-invariant
- and rotation-invariant

$\implies$  Simplest example of  $\gamma$ -SI and RI operators !

Theorem:  $\gamma$ -SI + RI  $\Leftrightarrow$  Laplacian [Th.3.39 of my report]

$(-\Delta)^{\gamma/2}$  is the **only**  $\gamma$ -SI and RI convolution

## ② Steerable operators

# Steerable convolution operators

**Definition** [Vonesch et al., 2015],[Unser and Chenouard, 2013]

$G$  is *steerable* when

$$\dim V_{\hat{g}} = \dim \operatorname{Span}_{\theta_0 \in [0, 2\pi]} \left\{ (r, \theta) \mapsto \hat{g}(r, \theta + \theta_0) \right\} \text{ is finite}$$



# Steerable convolution operators

**Definition** [Vonesch et al., 2015],[Unser and Chenouard, 2013]

$G$  is *steerable* when

$$\dim V_{\hat{g}} = \dim \operatorname{Span}_{\theta_0 \in [0, 2\pi]} \left\{ (r, \theta) \mapsto \hat{g}(r, \theta + \theta_0) \right\} \text{ is finite}$$

*Order of steerability*

$$n_G \stackrel{\text{def}}{=} \dim V_{\hat{g}} \in \mathbb{N}$$

Example: Null-operator  $G = 0 \Leftrightarrow n_G = 0$

# Steerability generalizes rotation-invariance

Theorem

[Th.4.22 of my report]

Non-zero steerable of order  $n_G = 1 \Leftrightarrow$  rotation-invariant

Just a sanity check ...

# First characterization of 2D steerable convolutions

## Theorem

Hard, [Vonesch et al., 2015]

$G$  is a 2D steerable convolution  $\Leftrightarrow$   
there exists an integer  $K_G$ , and functions  $\rho_k : \mathbb{R}_+ \rightarrow \mathbb{C}$  such that

$$\hat{g}(r, \theta) = \sum_{-K_G \leq k \leq K_G} \rho_k(r) e^{jk\theta}$$

# First characterization of 2D steerable convolutions

## Theorem

Hard, [Vonesch et al., 2015]

$G$  is a 2D steerable convolution  $\Leftrightarrow$   
there exists an integer  $K_G$ , and functions  $\rho_k : \mathbb{R}_+ \rightarrow \mathbb{C}$  such that

$$\hat{g}(r, \theta) = \sum_{-K_G \leq k \leq K_G} \rho_k(r) e^{jk\theta}$$

“Max frequency”  $K_G$

$K_G \in \mathbb{N}$  is unique for non-zero  $G$

# First characterization of 2D steerable convolutions

## Theorem

Hard, [Vonesch et al., 2015]

$G$  is a 2D steerable convolution  $\Leftrightarrow$   
there exists an integer  $K_G$ , and functions  $\rho_k : \mathbb{R}_+ \rightarrow \mathbb{C}$  such that

$$\hat{g}(r, \theta) = \sum_{-K_G \leq k \leq K_G} \rho_k(r) e^{jk\theta}$$

“Max frequency”  $K_G$

$K_G \in \mathbb{N}$  is unique for non-zero  $G$

Still too general !

The radial functions  $\rho_k(r)$  are completely **unspecified**

## ③ Scale-invariance for steerable convolutions

# Steerable scale-invariant convolutions

What does scale-invariance adds?

[Th.4.38 of my report]

$$\implies \rho_k(r) = a_k r^\gamma, \quad \forall k$$

# Steerable scale-invariant convolutions

What does scale-invariance adds?

[Th.4.38 of my report]

$$\implies \rho_k(r) = a_k r^\gamma, \quad \forall k$$

And so:  $G$  is  $\gamma$ -scale-invariant and steerable

$$\Leftrightarrow \hat{g}(r, \theta) = r^\gamma \sum_{-K_G \leq k \leq K_G} a_k e^{jk\theta}$$

$\implies$  **Separable form** between  $r$  and  $\theta$  : great !

Strong result with an easy proof

On the white board





# Steerable scale-invariant convolutions

What does scale-invariance adds?

[Th.4.38 of my report]

$$\implies \rho_k(r) = a_k r^\gamma, \quad \forall k$$

And so:  $G$  is  $\gamma$ -scale-invariant and steerable

$$\Leftrightarrow \hat{g}(r, \theta) = r^\gamma \sum_{-K_G \leq k \leq K_G} a_k e^{jk\theta}$$

$\implies$  **Separable form** between  $r$  and  $\theta$  : great !

Parametrization of steerable  $\gamma$ -SI convolutions

new !

With  $K_G \in \mathbb{N}$  and  $2K_G + 1$  complex parameters  $a_k = \rho_k(1)$

And their **polar** part is just a trigonometric polynomial (in  $e^{j\theta}$ ) !

# Hermitian-symmetric Fourier multiplier

Hermitian-symmetric  $\hat{g}$

[Stein and Weiss, 1971]

$G$  is real  $\Leftrightarrow \forall f, G\{f\}$  is real-valued  $\Leftrightarrow \hat{g}$  is *Hermitian-symmetric* :

$$\hat{g}(r, \theta + \pi) = \overline{\hat{g}(r, \theta)}$$

# Hermitian-symmetric Fourier multiplier

Hermitian-symmetric  $\hat{g}$  [Stein and Weiss, 1971]

$G$  is real  $\Leftrightarrow \forall f, G\{f\}$  is real-valued  $\Leftrightarrow \hat{g}$  is *Hermitian-symmetric* :

$$\hat{g}(r, \theta + \pi) = \overline{\hat{g}(r, \theta)}$$

Consequence on the coefficients  $(a_k)$  ?

$$a_{-k} = (-1)^k \overline{a_k}, \forall k$$

Parametrization

better !

$\implies$  With  $K_G \in \mathbb{N}$  and parameters  $\{a_0, \dots, a_{K_G}\}$  only

## Example with $n_G = 1$ : the fractional Laplacian

$G = (-\Delta)^{\gamma/2}$  is steerable, of order  $n_G = 1$

### Fourier multiplier

$\hat{g}(r, \theta) = r^\gamma$  is Hermitian-symmetric, RI and  $\gamma$ -SI

### With our parametrization

With  $K_G = 0$ , and  $a_0 = 1$

Check:  $a_0 = (-1)^0 \overline{a_0}$



## Example with $n_G = 2$ : directional derivatives

$G = D_\alpha$  is steerable, of order  $n_G = 2$

Fourier multiplier

[Chaudhury and Unser, 2010]

$\hat{g}(r, \theta) = jr \cos(\theta - \alpha)$  is Hermitian-symmetric, *not* RI and 1-SI

With our parametrization

With  $K_G = 1$ , and  $a_{-1} = e^{-j\alpha}/2$ ,  $a_0 = 0$ ,  $a_1 = e^{j\alpha}/2$

Check:  $a_{-1} = (-1)^1 \bar{a}_1$



## Example with $n_G = 2$ : directional derivatives

$G = D_\alpha$  is steerable, of order  $n_G = 2$

Fourier multiplier

[Chaudhury and Unser, 2010]

$\hat{g}(r, \theta) = jr \cos(\theta - \alpha)$  is Hermitian-symmetric, *not* RI and 1-SI

Sanity check

And it has  $n_G = 2$

Good: steerability is more general than rotation-invariant



## Example with $n_G = 3$ : the “Mondrian” $D_x D_y$

$G = D_x D_y$  is steerable, of order  $n_G = 3$

### Fourier multiplier

$\hat{g}(r, \theta) = -r^2 \cos(\theta) \sin(\theta)$  is Hermitian-sym., *not* RI and 2-SI

### With our parametrization

With  $K_G = 2$ , and  $a_{-2} = -j/4$ ,  $a_{-1} = a_0 = a_1 = 0$ ,  $a_2 = j/4$

Check:  $a_{-2} = (-1)^2 \overline{a_2}$



## ④ Decompositions of steerable convolutions



# First decomposition : with a trigonometric polynomial

Steerable  $\gamma$ -SI convolutions

[Th.4.38]

$$\hat{g}(r, \theta) = r^\gamma \sum_{-K_G \leq k \leq K_G} a_k e^{jk\theta}$$

Already interesting and useful ...

- Simple parametrization
- Easy to implement

# First decomposition : with a trigonometric polynomial

Steerable  $\gamma$ -SI convolutions


[Th.4.38]

$$\hat{g}(r, \theta) = r^\gamma \sum_{-K_G \leq k \leq K_G} a_k e^{jk\theta}$$

Already interesting and useful ...

- Simple parametrization
- Easy to implement

... But

 Sums are not easy to invert, if we want  $G^{-1}$  !

**Can we do better ?** *Yes we can !*

# Decomposition as a product, with $G_0$

Theorem: *Partly* factorized decomposition

[Th.4.49]

$G$  is a 2D steerable  $\gamma$ -scale-invariant convolutions

$$\Leftrightarrow G \propto (-\Delta)^{(\gamma-m)/2} \circ D_{\alpha_1} \circ \dots \circ D_{\alpha_m} \circ G_0$$

- $G_0$  : invertible and 0-SI
- With  $\hat{g}_0(r, \theta) \neq 0$ , a trigonometric polynomial of degree  $K_G - m$

# Decomposition as a product, with $G_0$

Theorem: *Partly* factorized decomposition

[Th.4.49]

$G$  is a 2D steerable  $\gamma$ -scale-invariant convolutions

$$\Leftrightarrow G \propto (-\Delta)^{(\gamma-m)/2} \circ D_{\alpha_1} \circ \dots \circ D_{\alpha_m} \circ G_0$$

- $G_0$  : invertible and 0-SI
- With  $\hat{g}_0(\mathbf{r}, \theta) \neq 0$ , a trigonometric polynomial of degree  $K_G - m$

Great !

- Now we can think of inverting  $\hat{g}(\mathbf{r}, \theta) \dots$
- If we know how to inverse  $(-\Delta)^{(\gamma-m)/2}$  and  $D_{\alpha_i} \dots$   
 $\implies$  We can use  $G^{-1}$

# Decomposition as a product, with $G_0$

## Sketch of the proof

Long, cf. p60-63 of my report

Work on the roots  $z_i \in Z_P$  of the complex trigonometric polynomial

$$P(e^{j\theta}) = e^{jK_G\theta} \sum a_k (e^{j\theta})^k = a_{K_G} \prod_{z_i} (e^{j\theta} - z_i)$$

By Hermitian symmetry :  $z_i \in Z_P \Leftrightarrow -1/\bar{z}_i \in Z_P$

Let  $z_i = r_i e^{j\beta_i}$

– 0 is not a root :  $r_i \neq 0$



# Decomposition as a product, with $G_0$

## Sketch of the proof

Long, cf. p60-63 of my report

Work on the roots  $z_i \in Z_P$  of the complex trigonometric polynomial

$$P(e^{j\theta}) = e^{jK_G\theta} \sum a_k (e^{j\theta})^k = a_{K_G} \prod_{z_i} (e^{j\theta} - z_i)$$

By Hermitian symmetry :  $z_i \in Z_P \Leftrightarrow -1/\bar{z}_i \in Z_P$

Let  $z_i = r_i e^{j\beta_i}$



- 0 is not a root :  $r_i \neq 0$
- Some roots are on the unit circle :  $r_i = 1$   
 $e^{j\beta_i}$  and  $-1/\overline{e^{j\beta_i}} = -e^{j\beta_i} = e^{j(\beta_i+\pi)}$

Group them by pairs, add a  $r$  term (and some computations)

$\Rightarrow$  gives a derivative  $D_{\beta_i+\frac{\pi}{2}}$

# Decomposition as a product, with $G_0$

## Sketch of the proof

Long, cf. p60-63 of my report

Work on the roots  $z_i \in Z_P$  of the complex trigonometric polynomial

$$P(e^{j\theta}) = e^{jK_G\theta} \sum a_k (e^{j\theta})^k = a_{K_G} \prod_{z_i} (e^{j\theta} - z_i)$$

By Hermitian symmetry :  $z_i \in Z_P \Leftrightarrow -1/\bar{z}_i \in Z_P$

Let  $z_i = r_i e^{j\beta_i}$



- 0 is not a root :  $r_i \neq 0$
- Some roots are on the unit circle :  $r_i = 1$   
 $e^{j\beta_i}$  and  $-1/\overline{e^{j\beta_i}} = -e^{j\beta_i} = e^{j(\beta_i+\pi)}$

Group them by pairs, add a  $r$  term (and some computations)

$\implies$  gives a derivative  $D_{\beta_i+\frac{\pi}{2}}$

- For the roots of modulus  $r_i \neq 1$ ,  $(e^{j\theta} - z_i)$  is non-canceling  
 $\implies$  collect all of them, in  $\hat{g}_0(r, \theta)$

# Decomposition as a product, with $G_0$

Theorem: *Partly* factorized decomposition

[Th.4.49]

$G$  is a 2D steerable  $\gamma$ -scale-invariant convolutions

$$\Leftrightarrow G \propto (-\Delta)^{(\gamma-m)/2} \circ D_{\alpha_1} \circ \dots \circ D_{\alpha_m} \circ G_0$$

- $G_0$  : invertible and 0-SI
- With  $\hat{g}_0(r, \theta) \neq 0$ , a trigonometric polynomial of degree  $K_G - m$

Great !

**But can we do *even* better ?** *Yes we can !*



# Decomposition as product of elementary blocks $G_{\lambda, \alpha}$

Can we do better ? *Sure !*

Finish the factorization of the trigonometric polynomial

Decomposition as product of elementary blocks  $G_{\lambda, \alpha}$ 

Can we do better ? *Sure !*

Finish the factorization of the trigonometric polynomial

Finishing the proof ...

The roots of modulus  $r_i \neq 1$ , can also be grouped by pairs :

$$(e^{j\theta} - z_i)(e^{j\theta} + 1/\bar{z}_i) = \dots$$

Let  $z_i = r_i e^{j\beta_i}$ , we obtain

$$\dots = 2j \cos(\theta - (\beta_i - \pi/2)) + \underbrace{(1/r_i - r_i)}_{\neq 0} \neq 0$$

Decomposition as product of elementary blocks  $G_{\lambda, \alpha}$ 

Can we do better? *Sure!*

Finish the factorization of the trigonometric polynomial

Finishing the proof ...

The roots of modulus  $r_i \neq 1$ , can also be grouped by pairs :

$$(e^{j\theta} - z_i)(e^{j\theta} + 1/\bar{z}_i) = \dots$$

Let  $z_i = r_i e^{j\beta_i}$ , we obtain

$$\dots = 2j \cos(\theta - (\beta_i - \pi/2)) + \underbrace{(1/r_i - r_i)}_{\neq 0} \neq 0$$

Adding a  $r$  gives a **convex combination** of  $D_{\alpha_i}$  and  $(-\Delta)^{1/2}$  :

$$r \left( 2j \cos(\theta - \alpha_i) + (1/r_i - r_i) \right) \propto \left( \lambda_i \widehat{D}_{\alpha_i}(r, \theta) + (1 - \lambda_i) \widehat{(-\Delta)^{1/2}}(r, \theta) \right)$$

Decomposition as product of elementary blocks  $G_{\lambda_i, \alpha_i}$ 

Theorem: Fully factorized decomposition [Th.5.1 of my report]

$G$  is a 2D steerable  $\gamma$ -scale-invariant convolution

$$\Leftrightarrow G \propto (-\Delta)^{(\gamma - K_G)/2} \circ \bigcirc_{i=1}^{K_G} \underbrace{(\lambda_i D_{\alpha_i} + (1 - \lambda_i)(-\Delta)^{1/2})}_{\stackrel{\text{def}}{=} G_{\lambda_i, \alpha_i}}$$

- With **convex** weights  $\lambda_1, \dots, \lambda_{K_G} \in (0, 1]$
- And angles  $\alpha_1, \dots, \alpha_{K_G} \in [0, 2\pi]$
- $G_{\lambda_i, \alpha_i}$  is a 1-SI convolution, steerable of order  $n_G \leq 2$  and  $K_G \leq 1$

# Decomposition as product of elementary blocks $G_{\lambda_i, \alpha_i}$

Theorem: Fully factorized decomposition [Th.5.1 of my report]

$G$  is a 2D steerable  $\gamma$ -scale-invariant convolution

$$\Leftrightarrow G \propto (-\Delta)^{(\gamma - K_G)/2} \circ \bigcirc_{i=1}^{K_G} \underbrace{(\lambda_i D_{\alpha_i} + (1 - \lambda_i)(-\Delta)^{1/2})}_{\stackrel{\text{def}}{=} G_{\lambda_i, \alpha_i}}$$

- With **convex** weights  $\lambda_1, \dots, \lambda_{K_G} \in (0, 1]$
- And angles  $\alpha_1, \dots, \alpha_{K_G} \in [0, 2\pi]$
- $G_{\lambda_i, \alpha_i}$  is a 1-SI convolution, steerable of order  $n_G \leq 2$  and  $K_G \leq 1$

## Natural interpretation

- $\alpha$  is the *direction* of the derivative
- $\lambda$  is a “*degree of directionality*”

Decomposition as product of elementary blocks  $G_{\lambda_i, \alpha_i}$ 

Theorem: Fully factorized decomposition [Th.5.1 of my report]

$G$  is a 2D steerable  $\gamma$ -scale-invariant convolution

$$\Leftrightarrow G \propto (-\Delta)^{(\gamma - K_G)/2} \circ \bigcirc_{i=1}^{K_G} \underbrace{(\lambda_i D_{\alpha_i} + (1 - \lambda_i)(-\Delta)^{1/2})}_{\stackrel{\text{def}}{=} G_{\lambda_i, \alpha_i}}$$

- With **convex** weights  $\lambda_1, \dots, \lambda_{K_G} \in (0, 1]$
- And angles  $\alpha_1, \dots, \alpha_{K_G} \in [0, 2\pi]$
- $G_{\lambda_i, \alpha_i}$  is a 1-SI convolution, steerable of order  $n_G \leq 2$  and  $K_G \leq 1$

Great !

- Simpler parametrization
- Now, we can more easily invert  $\hat{g}(r, \theta)$  (for  $r \neq 0$ )

## ⑤ Illustrations on sparse stochastic processes

# Computing $G_{\lambda,\alpha}^{-1}$ ?

Fourier multiplier of  $G_{\lambda,\alpha}^{-1}$  ?

Obvious, **but maybe not always well defined** :

$$\frac{1}{\widehat{g_{\lambda,\alpha}}(r, \theta)}$$



# Computing $G_{\lambda,\alpha}^{-1}$ ?

Fourier multiplier of  $G_{\lambda,\alpha}^{-1}$  ?

Obvious, **but maybe not always well defined** :

$$\frac{1}{\widehat{g_{\lambda,\alpha}}(\mathbf{r}, \boldsymbol{\theta})}$$

Impulse response of  $G_{\lambda,\alpha}^{-1}$  ?

$$\rho_{G_{\lambda,\alpha}} = \mathcal{F}^{-1} \left\{ \frac{1}{\widehat{g_{\lambda,\alpha}}(\mathbf{r}, \boldsymbol{\theta})} \right\}$$

- Known for Laplacians and derivatives ( $\lambda = 0$  or  $1$ ) ☺
- Harder for our “*partly directional*” block  $G_{\lambda,\alpha}$  ( $0 < \lambda < 1$ ) ☹

# Implementing $G_{\lambda,\alpha}^{-1}$

## Implementation

- In the discrete Fourier domain
- “Simple” point-wise **division** by

$$\widehat{g_{\lambda,\alpha}}[m, n] = r[m, n] \left( \lambda j \cos(\theta[m, n] - \alpha) + (1 - \lambda) \right)$$

- Not too hard to implement

With Virginie

# Implementing $G_{\lambda,\alpha}^{-1}$

## Implementation

- In the discrete Fourier domain
- “Simple” point-wise **division** by

$$\widehat{g_{\lambda,\alpha}}[m, n] = r[m, n] \left( \lambda \cos(\theta[m, n] - \alpha) + (1 - \lambda) \right)$$

- Not too hard to implement

With Virginie

## But ...

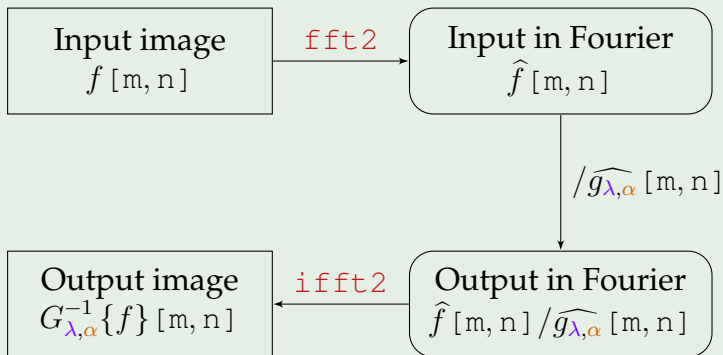
- Discretization errors :  $\text{fft2} \neq \mathcal{F}$  and  $\text{ifft2} \neq \mathcal{F}^{-1}$
- Our choice : if  $\widehat{g_{\lambda,\alpha}}[m, n] = 0$ , **force** it = 1

$$G\_inv(isinf(G\_inv)) = 1$$

$\Rightarrow$  **Approximations !**

# Implementing $G_{\lambda,\alpha}^{-1}$ with `fft2` and `ifft2`

Apply  $G_{\lambda,\alpha}^{-1}$  to a real 2D image  $f[m, n]$



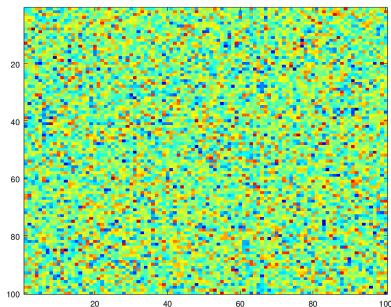
⚠ Small approximation error if  $\widehat{g_{\lambda,\alpha}}[m, n]$  can be zero !

# Gaussian and Poisson-Gaussian sparse processes

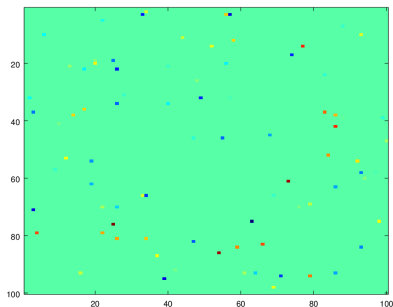
Quick reminders ...

[Unser and Tafti, 2014]

Two examples of realizations :



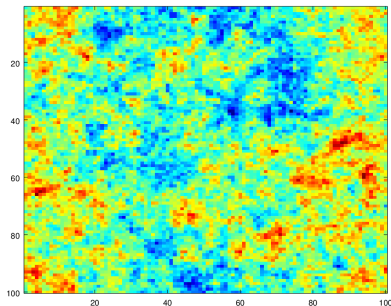
(a) Gaussian white noise  
(i.i.d. Gaussian on every pixels)



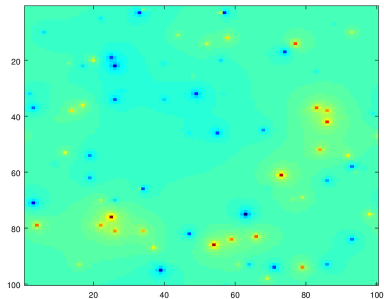
(b) Compound Poisson-Gaussian  
(i.i.d. Gaussian on the "firing" pixels)

# One block $G_{\lambda,0}^{-1}$ , increasing $\lambda$

Example #1 with  $\alpha = 0$ , and  $\lambda = 0, 0.25, 0.5, 0.75, 1$



(a) On a Gaussian white noise

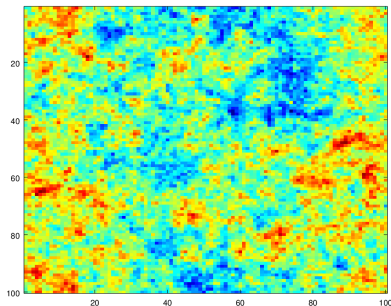


(b) On a "low-firing" Poisson

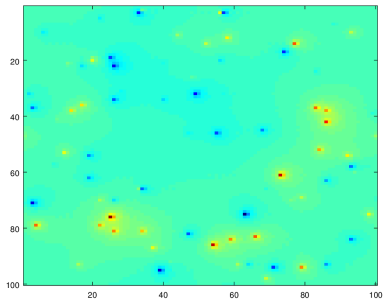
Figure:  $G_{0,0}^{-1}$  : purely isotropic

# One block $G_{\lambda,0}^{-1}$ , increasing $\lambda$

Example #1 with  $\alpha = 0$ , and  $\lambda = 0, 0.25, 0.5, 0.75, 1$



(a) On a Gaussian white noise

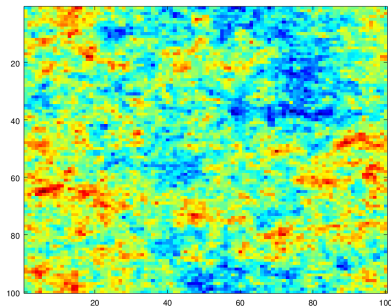


(b) On a "low-firing" Poisson

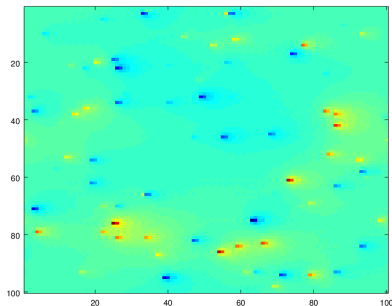
Figure:  $G_{0.25,0}^{-1}$  : not yet directional

# One block $G_{\lambda,0}^{-1}$ , increasing $\lambda$

Example #1 with  $\alpha = 0$ , and  $\lambda = 0, 0.25, 0.5, 0.75, 1$



(a) On a Gaussian white noise



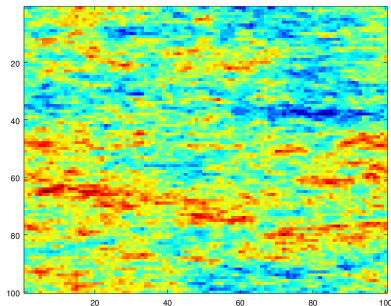
(b) On a "low-firing" Poisson

Figure:  $G_{0.50,0}^{-1}$  : not much directional

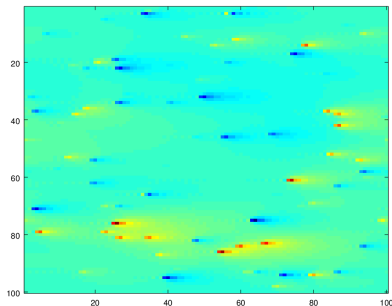


# One block $G_{\lambda,0}^{-1}$ , increasing $\lambda$

Example #1 with  $\alpha = 0$ , and  $\lambda = 0, 0.25, 0.5, 0.75, 1$



(a) On a Gaussian white noise

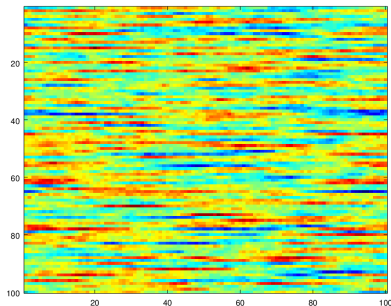


(b) On a "low-firing" Poisson

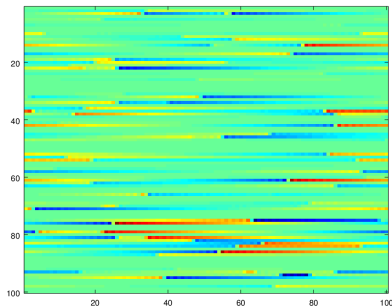
Figure:  $G_{0.75,0}^{-1}$  : more directional, along  $\rightarrow$

# One block $G_{\lambda,0}^{-1}$ , increasing $\lambda$

Example #1 with  $\alpha = 0$ , and  $\lambda = 0, 0.25, 0.5, 0.75, 1$



(a) On a Gaussian white noise

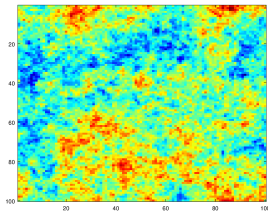


(b) On a "low-firing" Poisson

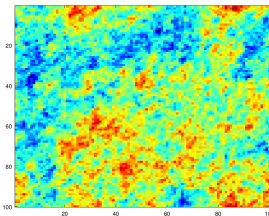
Figure:  $G_{1,0}^{-1}$ : purely directional, along  $\rightarrow$

# One block $G_{\lambda, \pi/6}^{-1}$

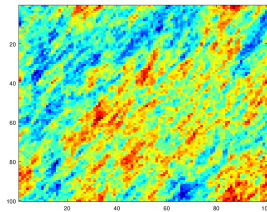
Example #2 with  $\alpha = \pi/6$  on a Gaussian white noise



(a)  $\lambda = 0.25$

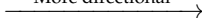


(b)  $\lambda = 0.5$



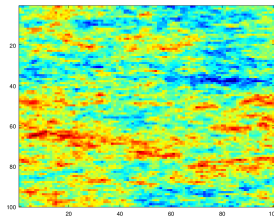
(c)  $\lambda = 0.75$

More directional

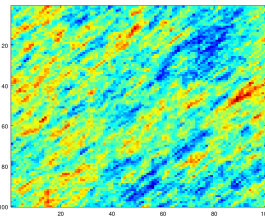


# One block $G_{0.75,\alpha}^{-1}$ , turning $\alpha$

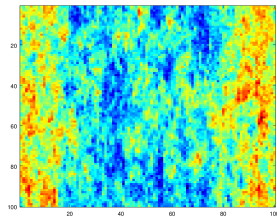
Example #3 with  $\lambda = 0.75$ , and  $\alpha = 0, \pi/4, \pi/2$  on a Gaussian



(a)  $\alpha = 0$



(b)  $\alpha = \pi/4$



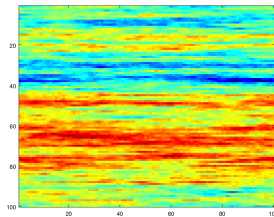
(c)  $\alpha = \pi/2$

Rotation by  $+\pi/4$  ↷  
 →

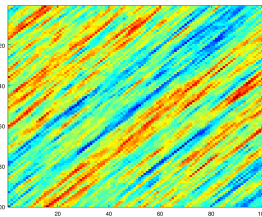
Very directional

# One block $G_{0.95, \alpha}^{-1}$ , turning $\alpha$

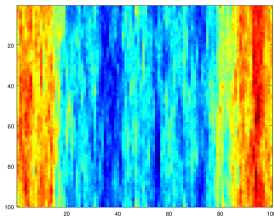
Example #4 with  $\lambda = 0.95$ , and  $\alpha = 0, \pi/4, \pi/2$  on a Gaussian



(a)  $\alpha = 0$



(b)  $\alpha = \pi/4$



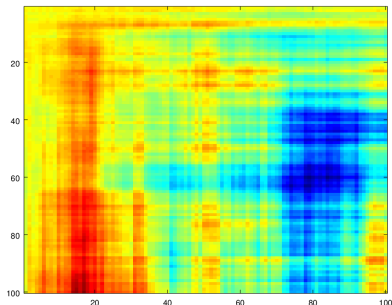
(c)  $\alpha = \pi/2$

Rotation by  $+\pi/4$  ↷  
 →

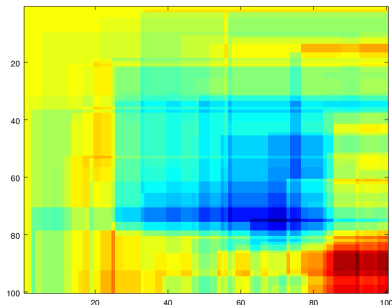
Almost purely directional

# Two *purely*-directional blocks, $G_{1,0}^{-1}G_{1,\pi/2}^{-1}$

Example #5 with the inverse “Mondrian” derivative  $D_x^{-1}D_y^{-1}$



(a) On a Gaussian white noise

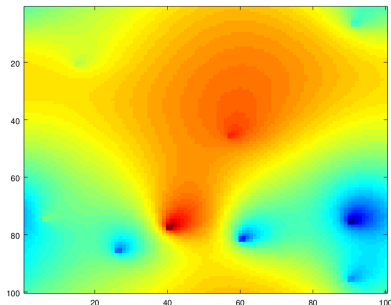


(b) On a “low-firing” Poisson

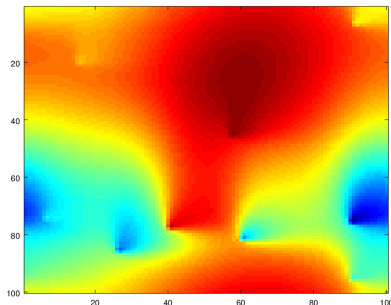
**Figure:** Purely directional, two orthogonal integrations = “Mondrian” process

# Two *partly*-directional blocks, $G_{\lambda,0}^{-1}G_{\lambda,\pi/2}^{-1}$

Example #6 with a partly-directional “Mondrian”



(a)  $\lambda = 0.5$

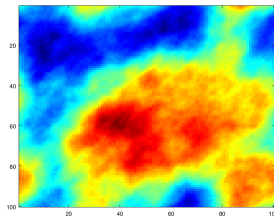


(b)  $\lambda = 0.75$

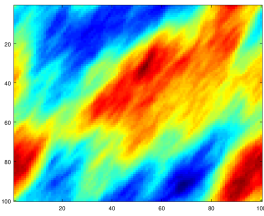
**Figure:** Partly directional “Mondrian” process, on a “low-firing” Poisson

# Two blocks, same angle, $G_{\lambda, \pi/3}^{-1} G_{\lambda, \pi/3}^{-1}$

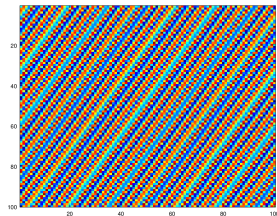
Example #7 with  $\alpha_1 = \alpha_2 = \pi/3$  on a Gaussian white noise



(a)  $\lambda = 0.5$

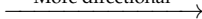


(b)  $\lambda = 0.75$



(c)  $\lambda = 1$

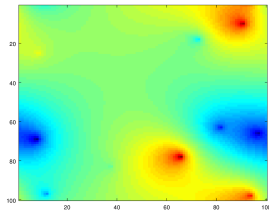
More directional



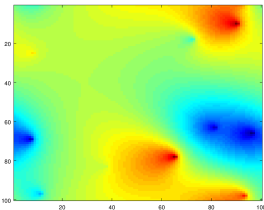


# Two blocks, two angles, $G_{\lambda, 3\pi/4}^{-1} G_{\lambda, 5\pi/4}^{-1}$

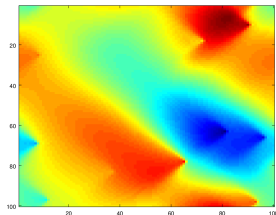
Example #8 with  $\alpha_1 = \frac{3\pi}{4}$ ,  $\alpha_2 = \frac{5\pi}{4}$  on a “low-firing” Poisson



(a)  $\lambda = 0.3$



(b)  $\lambda = 0.5$



(c)  $\lambda = 0.8$

More directional  $\rightarrow$

$\simeq$  Cone of opening  $\pi/2$ ,  
direction  $\pi$  (along  $\leftarrow$ )

## ⑥ Conclusion & Appendix

## Quick sum-up ...

### First we presented ...

- Convolution operators,  $G$
- Fourier multipliers,  $\hat{g}$
- Geometrical properties ( $\gamma$ -SI, RI) on  $G \Leftrightarrow$  on  $\hat{g}$
- And the notion of **steerability**

$$\dim V_{\hat{g}} = \dim \operatorname{Span}_{\theta_0 \in [0, 2\pi]} \left\{ (r, \theta) \mapsto \hat{g}(r, \theta + \theta_0) \right\} \text{ is finite}$$

## Quick sum-up ...

### Then we found and proved ...

- Characterization of 2D  $\gamma$ -SI steerable convolutions as a *sum*
- Parameters:  $K_G$  and  $\{a_0, \dots, a_{K_G}\}$ , but not used in practice
- And also a decomposition as a *product*, of simple blocks  $G_{\lambda, \alpha}$

$$G_{\lambda_i, \alpha_i} = \left( \lambda_i D_{\alpha_i} + (1 - \lambda_i) (-\Delta)^{1/2} \right)$$

- The blocks are *exactly* the 1-SI steerable of order  $n_G \leq 2$
- And are **convex combinations** of
  - a directional derivative  $D_{\alpha_i}$  (order 2)
  - and the half-Laplacian  $(-\Delta)^{1/2}$  (order 1)

## Quick sum-up ...

### And experimentally, we applied ...

On a Gaussian white noise and a compound Poisson noise :

- Purely isotropic  $(-\Delta)^{\gamma/2}$  ( $\lambda = 1$ )
  - Or purely directional  $D_\alpha$  ( $\lambda = 0$ )
  - And *partly* directional  $G_{\lambda,\alpha}$  ( $\lambda \in (0, 1)$ )
  - And two blocks  $G_{\lambda,\alpha}$ , with different  $\lambda$  and  $\alpha$
- $\implies$  Interesting patterns ...

## Future work ...

### For the theoretical part ...

- More general theorem of decomposition (general case for  $\hat{g} \in \mathbf{S}'(\mathbb{R}^2)$ )
- Study steerability for higher dimensions  $d > 2$  ? (harder)

# Future work ...

## Applications to sparse processes ...

- Visualize our operators
  - But also generate new processes
- ⇒ Future publication with Julien and Virginie

## Other possibilities of applications, in the lab

- Generate new splines ...  
(an operator  $G \Leftrightarrow$  a spline)
- New data recovery algorithms ...  
(an operator  $G \Leftrightarrow$  a penalization term  $\|Gf\|$ )

Thank you !

Thank **you** for your attention !

... and  $\infty$  thanks to all of you for the last 4 months !! ☺



# Questions ?

# Questions ?

## Want to know more ?

↪ Read my master thesis / internship report :

<https://goo.gl/xPzw4A>

↪ And e-mail me if needed :

[lilian.besson@ens-cachan.fr](mailto:lilian.besson@ens-cachan.fr)

↪ Or consult the references

# Appendix

## Outline of the appendix

- Some proofs
- Main references given below
- Code, figures, results from our experiments, etc :  
→ [lbo.k.vu/epf12016](http://lbo.k.vu/epf12016)
- Everything here is open-source, under the *CC-BY 4.0 License*

# Proof of: steerability of order 1 = rotation-invariance

Theorem

[Th.4.22 of my report]

Non-zero steerable of order  $n_G = 1 \Leftrightarrow$  rotation-invariant

Quick proof

$\Leftarrow$  Obvious :  $V_{\hat{g}} = \text{Span}\{\hat{g}\} = \mathbb{R}\hat{g}$ , has dimension  $n_G = 1$

# Proof of: steerability of order 1 = rotation-invariance

## Theorem

[Th.4.22 of my report]

Non-zero steerable of order  $n_G = 1 \Leftrightarrow$  rotation-invariant

## Quick proof

$\Rightarrow$  Less obvious but not too hard :

- $n_G = 1$  so there exists  $\lambda(\theta_0) \in \mathbb{R}$  such that  $R_{\theta_0}\{\hat{g}\} = \lambda(\theta_0)\hat{g}$ ,
- $R_{\theta_0} : (r, \theta) \mapsto (r, \theta + \theta_0)$  is a bijective change of variable,
- so  $R_{\theta_0}\{\hat{g}\}$  and  $\hat{g}$  have same  $L^2$  norm on the circle  $\theta \in [0, 2\pi]$  (for a fixed  $r$ ),
- and so  $\lambda(\theta_0) = \pm 1$ .
- But  $R_{\theta_0} = R_{\theta_0/2}R_{\theta_0/2}$  so  $\lambda(\theta_0) = \lambda(\theta_0/2)^2 > 0$ ,
- and so finally  $R_{\theta_0}\{\hat{g}\} = \hat{g}$ .

So  $n_G = 1 \implies G$  is rotation-invariant as wanted.

# Previous works and references I

 Chaudhury, K. and Unser, M. (2009).

The fractional Hilbert transform and Dual-Tree Gabor-like wavelet analysis.




*In 2009 IEEE International Conference on Acoustics, Speech and Signal Processing*, pages 3205–3208. IEEE.

 Chaudhury, K. and Unser, M. (2010).

On the Shiftability of Dual-Tree Complex Wavelet transforms.

*IEEE Transactions on Signal Processing*, 58(1):221–232.

## Previous works and references II

-  Rudin, W. (1991).  
*Functional Analysis*.  
McGraw-Hill, Inc., New York.
-  Stein, E. M. and Weiss, G. L. (1971).  
*Introduction to Fourier analysis on Euclidean spaces*, volume 1.  
Princeton University Press.
-  Unser, M. and Chenouard, N. (2013).  
A unifying parametric framework for 2D steerable wavelet transforms.  
*SIAM Journal on Imaging Sciences*, 6(1):102–135.

## Previous works and references III

-  Unser, M., Fageot, J., and Ward, J. (2016).  
Splines are Universal Solutions of Linear Inverse Problems with Generalized-TV Regularization.  
*arXiv preprint arXiv:1603.01427.*
-  Unser, M. and Tafti, P. (2014).  
*An Introduction to Sparse Stochastic Processes.*  
Cambridge University Press.
-  Vonesch, C., Stauber, F., and Unser, M. (2015).  
Steerable PCA for Rotation-Invariant Image Recognition.  
*SIAM Journal on Imaging Sciences*, 8(3):1857–1873.



# Open-Source

## License

These slides and the report<sup>a</sup> are open-sourced under the *CC-BY 4.0 License*

Copyright 2016, © Lilian Besson

---

<sup>a</sup>And the additional resources – including code, figures, etc.