

MA 101: Key Concepts and Results

Vijaysekhar Chellaboina

Professor
Mahindra École Centrale

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Real Numbers \mathbb{R}

Basics

- ▶ Operations: $+$, \times , $-$, \div , identities: $0, 1$
- ▶ Relational: $<$, \leq , $>$, \geq
- ▶ Concepts: bounded, max, sup, inf, min, absolute value
- ▶ Subsets: Integers \mathbb{N} , rationals \mathbb{Q} , and irrationals $\mathbb{R} \setminus \mathbb{Q}$

Key results

- ▶ Every two numbers will satisfy either $x < y$, $y < x$, or $x = y$
- ▶ Every upper bounded set has a supremum
- ▶ Rationals are dense in \mathbb{R} ; so are irrationals
- ▶ $x^2 = 2$ has solutions in $\mathbb{R} \setminus \mathbb{Q}$; No solution in \mathbb{R} for $x^2 + 1 = 0$

Complex Numbers \mathbb{C}

Basics

- ▶ Starts with i , the solution to $x^2 + 1 = 0$
- ▶ Operations: $+$, \times , $-$, \div ; identities: 0 , 1
- ▶ No relational operators
- ▶ Concepts: Modulus/absolute value, argument/phase
- ▶ Subsets: Real numbers \mathbb{R} , Imaginary numbers

Key results

- ▶ $z = |z|e^{i\theta}$ where $\theta = \arg(z)$ (what is $\arg(z)$?)
- ▶ Euler's identity: $e^{i\theta} = \cos(\theta) + i \sin(\theta)$
- ▶ If $z^n = 1$ then $z^n = e^{i2k\pi}$, $k \in \mathbb{N}$. Hence, $z = e^{i\frac{2k\pi}{n}}$, $k \in \mathbb{N}$

Sequences

Basics

- ▶ A sequence is a function $f : \mathbb{N} \rightarrow \mathbb{R}$ or \mathbb{C}
- ▶ Concepts: Limit, convergence, divergence
- ▶ $\lim_{n \rightarrow \infty} a_n = L$: For every $\varepsilon > 0$, $\exists N$ s.t. $|a_n - L| < \varepsilon$, $n \geq N$
- ▶ $a_n \rightarrow \infty$: For every $M > 0$, $\exists N$ s.t. $a_n > M$, $n \geq N$.

Key results

- ▶ A convergent sequence is bounded; Converse, not true: $(-1)^n$
- ▶ An alternating sequence is divergent: $(-1)^n$, i^n , $\sin(n)$
- ▶ A monotonic sequence is convergent iff it is bounded
- ▶ Don't forget the squeeze theorem!

Series

Basics

- ▶ A series is a sequence $\sum_{k=1}^n a_n$
- ▶ Concepts: Limit, convergence, divergence

Key results

- ▶ If a nonnegative series converges then $a_n \rightarrow 0$ as $n \rightarrow \infty$
- ▶ Converse is not always true: $\sum_{k=1}^n \frac{1}{n}$
- ▶ But $\sum_{k=1}^n \frac{1}{n^p}$ converges for all $p > 1$
- ▶ Ratio, root tests can be quite handy!
- ▶ Don't forget the comparison test!

Remarks on Sequences and Series

▶ If $a_n \leq b_n$ for all n then $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$

▶ If $\lim_{n \rightarrow \infty} a_n$ and $\lim_{n \rightarrow \infty} b_n$ exist then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

▶ But $\lim_{n \rightarrow \infty} (a_n + b_n)$ may exist even otherwise

▶ For example, $s_n = \sum_{k=1}^n \frac{1}{k(k+1)}$ is convergent

▶ Proof: $0 \leq \frac{1}{k(k+1)} \leq \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

▶ Hence, s_n is bounded and monotonic \implies convergence

▶ However, $\sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^n \frac{1}{k+1}$

▶ where both $\sum_{k=1}^n \frac{1}{k}$ and $\sum_{k=1}^n \frac{1}{k+1}$ are divergent

Continuous Functions

Basics

- ▶ f is continuous at x if $\lim_{h \rightarrow 0} f(x+h) = f(x)$
- ▶ Same as: $\lim_{h \rightarrow 0^+} f(x+h) = \lim_{h \rightarrow 0^-} f(x+h) = f(x)$
- ▶ Polynomials, trigonometrics, exponential, logarithmics

Key results

- ▶ Sums, products, composites of continuous functions are continuous on domain of intersection
- ▶ Bolzano's theorem: if $f(a)f(b) < 0$ then $f(c) = 0$ for some $c \in (a, b)$
- ▶ Extreme value theorem: continuous functions on closed intervals have maximum and minimum

Convex Functions

Basics

- ▶ Let \mathcal{I} be an interval (closed, open, or semi-open)
- ▶ $f : \mathcal{I} \rightarrow \mathbb{R}$ is convex if for all $x, y \in \mathcal{I}$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

- ▶ Examples: $f(x) = x^2$ on \mathbb{R} , $-\ln(x)$ on $[0, \infty)$

Key results

- ▶ If f has a local minimum at x then it is a global minimum
- ▶ A convex f is continuous on the interior of \mathcal{I}
- ▶ If $f''(x) \geq 0 \forall x \in \mathcal{I}$ then f is convex

Differentiable Functions I

- ▶ f is differentiable at x if $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ exists
- ▶ Denoted by f' , it is the derivative of f
- ▶ If f' exists at x then f is continuous at x
- ▶ Chain rule 1: $(fg)' = f'g + fg'$
- ▶ Chain rule 2: $(f \circ g)' = (f' \circ g) \cdot g'$
- ▶ Local minimum: if f has a local minimum at x then $f'(x) = 0$ and $f''(x) \geq 0$
- ▶ Partial converse: if at x , $f'(x) = 0$ and $f''(x) > 0$ then f has a local minimum at x

Differentiable Functions II

Key Results

- ▶ Rolle's thm: If $f(a) = f(b)$ then $f'(c) = 0$ for some $c \in (a, b)$
- ▶ Mean value theorem: $\exists c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- ▶ Taylor series formula: there exists $c \in (x, x + h)$ such that

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(c)$$

- ▶ Anti-derivative: If $g = f'$ then g is the anti-derivative of f

Integrals I

- ▶ Area under a curve; area of shapes other than polygons
- ▶ Partitions, Riemann sums, integrability
- ▶ Bounded functions are integrable on finite intervals
- ▶ Infinite interval case is a bit more tricky
- ▶ Since continuous functions on finite intervals are bounded they are integrable
- ▶ FTC I: If $F(x) = \int_a^x f(t)dt$ then $F'(x) = f(x)$
- ▶ FTC II: If G is anti-derivative of g then

$$\int_a^b g(t)dt = G(b) - G(a)$$

Integrals II

- ▶ Leibnitz formula:

$$\frac{d}{dx} \int_a^{g(x)} f(t) dt = f(g(x))g'(x)$$

- ▶ Change of variables:

$$\int_{g(a)}^{g(b)} f(t) dt = \int_a^b f(g(u))g'(u) du$$

- ▶ Integration by parts:

$$\int f(t)g'(t) dt = f(t)g(t) - \int f'(t)g(t) dt + C$$

Differential Equations I

- ▶ Consider the 1st order ODE: $y' + P(x)y = Q(x)$
- ▶ Solutions exists and unique if P and Q are continuous on \mathcal{I}
- ▶ Homogeneous solution: $y(x) = y(a)e^{-A(x)}$ where $a \in \mathcal{I}$ and $A(x) = \int_a^x P(t)dt$
- ▶ Nonhomogeneous solution:

$$y(x) = y(a)e^{-A(x)} + e^{-A(x)} \int_a^x Q(t)e^{A(t)} dt$$

Differential Equations II

- ▶ Consider the 2nd order ODE: $y'' + ay' + by = R(x)$
- ▶ Characteristic equation: $r^2 + ar + b = 0$ with roots r_1, r_2
- ▶ Complementary (homogeneous) solution:
 - ▶ $y(x) = C_1e^{r_1x} + C_2e^{r_2x}$ if $r_1 \neq r_2$ and real
 - ▶ $y(x) = C_1e^{\alpha x} \cos(\beta x) + C_2e^{\alpha x} \sin(\beta x)$ if $r_{1,2} = \alpha \pm i\beta$
 - ▶ $y(x) = C_1e^{rx} + C_2xe^{rx}$ if $r_1 = r_2 = r$
- ▶ Nonhomogeneous solution:
 - ▶ methods of undetermined coefficients (good for polynomials/exponentials)
 - ▶ variation of parameters (general but tedious)

Vector Calculus

- ▶ Let $\vec{r}(t) = (x(t), y(t))$ denote a curve
- ▶ Length of a curve:

$$\int_a^b \|\vec{r}'(t)\| dt$$

- ▶ Of course, $\|\vec{r}'(t)\| = \sqrt{x'^2(t) + y'^2(t)}$
- ▶ Let $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$ denote position, velocity, and acceleration, respectively
- ▶ The curvature

$$\kappa(t) = \frac{\|\vec{a}(t) \times \vec{v}(t)\|}{\|\vec{v}(t)\|^3}$$