

## Exercise

Recall that  $\mathbb{R}_+^* \stackrel{\text{def}}{=} (0, +\infty)$  is the set of positive real numbers (i.e.  $\mathbb{R}_+^* = \{x : x \in \mathbb{R}, x > 0\}$ ). Let  $a, b \in \mathbb{R}_+^*$  ( $a > 0, b > 0$ ).

Consider the set  $E = \left\{ (-1)^n a + \frac{b}{n} : n \in \mathbb{N}^* \right\}$ . (Of course  $E \subset \mathbb{R}$ .)

1. (for A group) Determine (if they exist) the **maximum**, the **supremum**, and *one upper bound* of this set  $E$ .
2. (for B group) Determine (if they exist) the **minimum**, the **infimum**, and *one lower bound* of this set  $E$ .

## Solution for A group (max)

**Just to check:** First of all, in the definition of  $E$ ,  $n > 0$  because in  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ , so  $\frac{b}{n}$  is well defined.

**Studying  $E$ :** Then, for any  $n \in \mathbb{N}^*$ , let  $u_n = (-1)^n a + \frac{b}{n}$ .

We have  $u_{2n} = a + \frac{b}{2n}$  and  $u_{2n-1} = -a + \frac{b}{2n-1}$ .

The two sequences  $(u_{2n})_{n \in \mathbb{N}^*}$  and  $(u_{2m-1})_{m \in \mathbb{N}^*}$  are decreasing, because  $b > 0$ .

So if we define  $E_{\text{even}} \stackrel{\text{def}}{=} \{u_{2n} : n \in \mathbb{N}^*\}$  and  $E_{\text{odd}} \stackrel{\text{def}}{=} \{u_{2m-1} : m \in \mathbb{N}^*\}$ , then<sup>1</sup>  $E = E_{\text{even}} \cup E_{\text{odd}}$ . But because the sequences decrease, both  $E_{\text{even}}$  and  $E_{\text{odd}}$  have a greatest element:

- $\max E_{\text{even}} = u_2$  is well defined (for  $n = 1$ ), i.e.  $\max E_{\text{even}} = a + \frac{b}{2}$ ,
- and  $\max E_{\text{odd}} = u_1$  is well defined (for  $m = 1$ ), i.e.  $\max E_{\text{odd}} = -a + \frac{b}{1} = b - a$ .

Therefore  $E$  also have a greatest element, given by  $\max E = \max(\max(E_{\text{even}}), \max(E_{\text{odd}}))$ .

We conclude with  $\max E = \max(a + \frac{b}{2}, b - a)$ . So  $E$  is upper-bounded and  $\sup E = \max E$ .

One upper bound is  $a + b$ , and another one (less precise) is  $a + b + 2014$  (indeed,  $\forall x \in E, x \leq a + b$  so  $\forall x \in E, x \leq a + b + 2014$ ). Any number bigger than  $a + b$  is **an** upper bound!

**Bonus:** We precise the notation with  $E_{a,b} \stackrel{\text{def}}{=} \left\{ (-1)^n a + \frac{b}{n} : n \in \mathbb{N}^* \right\}$ .

Find one pair  $a, b$  such that  $\max E_{a,b} = a + \frac{b}{2}$  and another pair  $a', b'$  such that  $\max E_{a',b'} = b' - a'$ .

**Answer:** With  $a = 100, b = 1$ , we have  $\max E_{100,1} = a + \frac{b}{2} = 100 + \frac{1}{2}$ . With  $a' = 2, b' = 50$ , we have  $\max E_{2,50} = b' - a' = 50 - 2 = 48$ .

<sup>1</sup>In fact,  $E = E_{\text{even}} \cup E_{\text{odd}}$  : these two sets form a partition of  $E$ .

## Solution for B group (min)

**Just to check:** First of all, in the definition of  $E$ ,  $n > 0$  because in  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ , so  $\frac{b}{n}$  is well defined.

**Studying  $E$ :** Then, for any  $n \in \mathbb{N}^*$ , let  $u_n = (-1)^n a + \frac{b}{n}$ .

We have  $u_{2n} = a + \frac{b}{2n}$  and  $u_{2n-1} = -a + \frac{b}{2n-1}$ .

The two sequences  $(u_{2n})_{n \in \mathbb{N}^*}$  and  $(u_{2m-1})_{m \in \mathbb{N}^*}$  are decreasing, because  $b > 0$ .

And we know that  $u_{2n} \xrightarrow{n \rightarrow +\infty} a$  and  $u_{2m-1} \xrightarrow{m \rightarrow +\infty} -a$ , because  $\frac{b}{k} \xrightarrow{k \rightarrow +\infty} 0$ .

So if we define  $E_{\text{even}} \stackrel{\text{def}}{=} \{u_{2n} : n \in \mathbb{N}^*\}$  and  $E_{\text{odd}} \stackrel{\text{def}}{=} \{u_{2m-1} : m \in \mathbb{N}^*\}$ , then<sup>2</sup>  $E = E_{\text{even}} \cup E_{\text{odd}}$ .

But because the sequences decrease (strictly) so:

- $\inf E_{\text{even}} = \lim_{n \rightarrow +\infty} u_{2n} = a$ , and  $E_{\text{even}}$  does not have a smallest element,
- $\inf E_{\text{odd}} = \lim_{m \rightarrow +\infty} u_{2m-1} = -a$ , and  $E_{\text{odd}}$  does not have a smallest element.

And because of the partition<sup>2</sup>,  $\inf E = \min(\inf(E_{\text{even}}), \inf(E_{\text{odd}}))$ . We conclude by saying that  $E$  is lower-bounded and  $\inf E = \min(a, -a) = -a$  (because  $a > 0$ ), and  $E$  does not have a smallest element.

Therefore one lower bound is  $-a$ , and another one (less precise) is  $-a - 7$  (indeed,  $\forall x \in E, x \geq -a$  so  $\forall x \in E, x \geq -a - 7$ ). Any number smaller than  $-a$  is a lower bound!

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<sup>2</sup> In fact,  $= E_{\text{even}} \uplus E_{\text{odd}}$  : these two sets form a partition of  $E$ .