

## Sets and Applications.

1. Verify that these boolean formulas are *tautologies* (*i.e.* assertions that are **true**s no matter the truth values given to the variables  $A$ ,  $B$  and  $C$ ):

- (i)  $\text{not } (A \text{ and } B) \Leftrightarrow (\text{not } A) \text{ or } (\text{not } B)$  (first DE MORGAN law),
- (ii)  $\text{not } (A \text{ or } B) \Leftrightarrow (\text{not } A) \text{ and } (\text{not } B)$  (second DE MORGAN law),
- (iii)  $(A \text{ and } (B \text{ or } C)) \Leftrightarrow (A \text{ and } B) \text{ or } (A \text{ and } C)$  (first distributivity),
- (iv) ( extbfBonus: )  $(A \text{ or } (B \text{ and } C)) \Leftrightarrow (A \text{ or } B) \text{ and } (A \text{ or } C)$  (second distributivity),

2. Transform these sentences into mathematical properties, with the quantifiers  $\exists$  and  $\forall$ :

- (i) For any real number  $x$ ,  $x$  squared is non-negative,
- (ii) If a real number is non-positive, we get a non-negative number when we multiply it by  $-2$ ,
- (iii) For any complex number  $z$ , there exists a complex number  $y$  such that  $y^2 - 3y = z$ ,
- (iv) ( extbfBonus: ) If a real number has a square root, then it is non-negative.

3. Find a relationship between the two sets  $A$  and  $B$  for every of these cases: (ot might be inclusion, equality, ... ):

- (i)  $A = \mathbb{R}$  et  $B = \{x \in \mathbb{R} : x^2 + x + 1 \leq 0\}$ ,
- (ii)  $A = \mathbb{R}_+^*$  et  $B = \{x \in \mathbb{R} : 2x \geq x\}$ ,
- (iii)  $A = \{2x : x \in C\}$  et  $B = \{20x : x \in \mathbb{R}\}$  with  $C \stackrel{\text{def}}{=} \mathbb{R}_+$ ,
- (iv)  $A = \{2x : x \in C\}$  et  $B = \{20x : x \in \mathbb{R}\}$  with  $C \stackrel{\text{def}}{=} \mathbb{N}$ .

4. Write the negation of these statements, and try to determine which one is true (the initial formula or its negative form):

- (i)  $\forall x \in \mathbb{R}, x^2 + x + 1 \geq 0$ ,
- (ii)  $\exists y \in \mathbb{R}, (y \geq 0) \text{ and } (y \leq 0)$ ,
- (iii)  $\forall x \in \mathbb{R}, (x \geq 0 \stackrel{\text{def}}{=} (\exists y \in \mathbb{R}_+^*, x = y^2))$ ,
- (iv) (Bonus)  $\forall \forall \varepsilon \in \mathbb{R}_+^*, \exists \mu_{\forall \varepsilon} \in \mathbb{R}_+^*, \forall x \in \mathbb{R}, (|x - 1| \leq \mu_{\forall \varepsilon} \Rightarrow |x^2 - 1| \leq \forall \varepsilon)$  (this one is hard).

5. Prove<sup>1</sup> the following statements, for any  $n \in \mathbb{N}$ :

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<sup>1</sup>One could reason by induction on the integer  $n \geq 0$ .

$$\begin{aligned} \text{(i)} \quad & \sum_{k=0}^{k=n} k = \frac{n(n+1)}{2}, \\ \text{(ii)} \quad & \sum_{k=0}^{k=n} k^2 = \frac{n(n+1)(2n+1)}{6}, \\ \text{(iii)} \quad & \sum_{k=0}^{k=n} k^3 = \left(\frac{n(n+1)}{2}\right)^2. \end{aligned}$$