

### Ex. 1 (B group): Corner Circle

A unit circle is placed against a right angle. What is radius of the smaller circle (in cm) ?

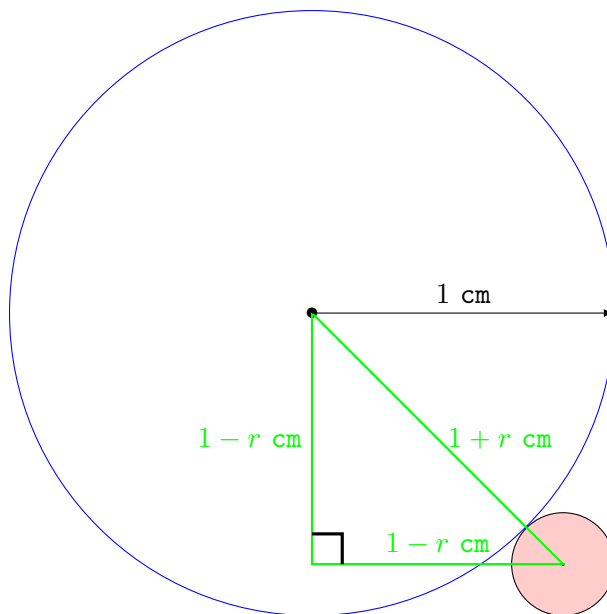


Figure 1: Corner Circle

**Solution:** We have  $(1+r)^2 = 2(1-r)^2$ , from the PYTHAGOREAN theorem (in the rectangle triangle in Figure 1). Then  $r$  is a solution of the equation  $r^2 - 6r + 1 = 0$ , so  $r = 3 \pm 2\sqrt{2}$ . But  $r < 1$ , then  $r = 3 - 2\sqrt{2} \simeq 0.1715$  cm.

### Ex. 2 (B group): Cubes and Multiples of 7

Prove that for any number that is not a multiple of seven, its cube is one more or one less than a multiple of seven. Formally, this would be written as:  $\forall n \in \mathbb{N}^*, 7 \nmid n \stackrel{\text{def}}{\implies} 7 \mid n^3 \pm 1$ .

**Solution:** Let write  $n$  as  $7a + b$ , with  $b \in \{1, 2, 3, 4, 5, 6\}$  the remainder of  $n$  divided by 7. We have  $n^3 = (7a + b)^3 = 7(7^2 \times a^3 + 3 \times 7a^2 \times b + 3 \times a \times b^2) + b^3 = 7c + b^3$  (with some  $c$ ), therefore  $n^3 = b^3 \pmod{7}$ . Therefore, we just have to compute  $b^3 \pmod{7}$  for all the 6 possibles values for  $b$ :

$b$	1	2	3	4	5	6
$b^3$	1	8	27	64	125	216
$b^3 \pmod{7}$	1	1	6	1	6	6

And  $6 = -1 \pmod{7}$ , so  $n^3 = \pm 1 \pmod{7}$

### Ex. 3 (B group): About continuity

Let  $f$  be a function from an interval  $[a, b]$  to  $\mathbb{R}$  (with  $a, b \in \mathbb{R}, a < b$ ), and  $x_0 \in [a, b]$ .

1. Recall the *formal* definition of  $f$  being continuous at this point  $x_0$  (as a pure formula with variables, well introduced with quantifiers  $\forall$  and  $\exists$ ).

2. If  $f$  is continuous at  $x_0$ , does this mean  $f$  is differentiable at  $x_0$ ?

If yes, (try to) prove it. If not, (try to) give a counter-example.

**Solution:**

1.  $f$  is continuous at  $x_0$  **iff**  $\forall \varepsilon > 0, (\exists \eta_{\varepsilon, x_0} > 0, (\forall y \in [a, b], (|x - y| < \eta_{\varepsilon, x_0} \stackrel{\text{def}}{=} |f(x_0) - f(y)| < \varepsilon)))$ ,
2. No, with the simple example of the absolute value  $|\cdot| : \mathbb{R} \longleftrightarrow \mathbb{R}, x \mapsto |x|$ , continuous on  $\mathbb{R}$ , but not differentiable on  $x_0 \stackrel{\text{def}}{=} 0$ .

### Ex. 1 (A group): 3 prime numbers?

Given that  $a, b, c \in \mathbb{N}^*$  are three non-negative integers, if we have  $abc + ab + bc + ca + a + b + c + 1 = 1001$ , what is the value of  $a + b + c$ ? Hint:  $1001 = 7 \times 11 \times 13$ .

**Solution:** We write  $abc + ab + bc + ca + a + b + c + 1$  as  $(a + 1)(b + 1)(c + 1)$ , and then we compute the prime decomposition of this number 1001 (given as a hint). The three integers are such that  $a > 0, b > 0, c > 0$ , we know that  $(a + 1) \geq 2, (b + 1) \geq 2, (c + 1) \geq 2$ . As we have  $1001 = 7 \times 11 \times 13 = (a + 1)(b + 1)(c + 1)$ , then by uniqueness of the prime decomposition, we can say that  $\{a + 1, b + 1, c + 1\} = \{7, 11, 13\}$  (as a non-ordered set). Finally,  $a + b + c = (a + 1) + (b + 1) + (c + 1) - 3 = 7 + 11 + 13 - 3 = 6 + 10 + 12 = 28$ .

### Ex. 2 (A group): Cables

A telephone company places round cables in round ducts.

Assuming the diameter of a cable is 2 cm, what would the diameter of the duct be for two cables, three cables and four cables?

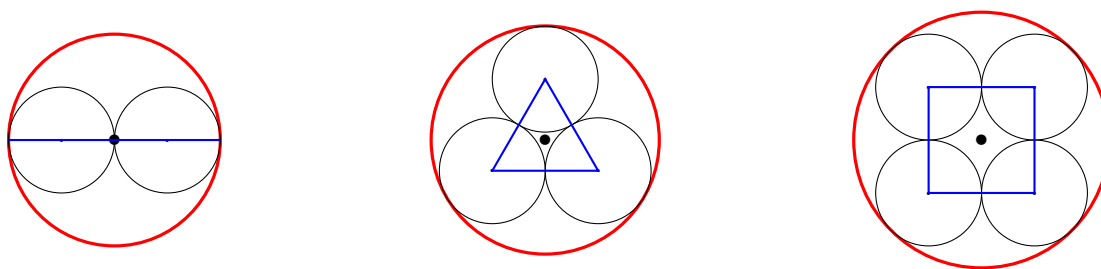


Figure 2: Cables

**Solution:** Quick version: 2 cm for two cables,  $2 + 4 \frac{\sqrt{3}}{3} \simeq 4.31$  cm for three cables, and  $\sqrt{8} + 2$  cm for four cables.

Clearly the duct for two cables must have a diameter of 4 cm, however, the duct for three and four cables requires a little more effort.

Let us first consider the three cable duct. The center of each circle form an equilateral triangle, of side of length 2 cm and of height  $h$ . First of all we use the PYTHAGOREAN theorem to find the height of the triangle,  $2^2 = h^2 + 1^2$ , therefore  $h = \sqrt{3} = 1.73$  cm. Using the geometric result that states the centre of an equilateral triangle is  $1/3$  the height of the triangle, we deduce that the distance from the centre of the triangle to its apex is  $(2/3)h = 2\sqrt{3}/3 \simeq 1.15$  cm. So the duct radius will be  $2\sqrt{3}/3 + 1$ , hence the diameter will be  $4\sqrt{3}/3 + 2$  cm.

Now we consider the duct for four cables. The center of each circle form a square, of side of length 2 cm, and of diagonal  $d$ . Using the PYTHAGOREAN theorem,  $d^2 = 2^2 + 2^2$ , giving  $d = \sqrt{8} \simeq 2.83$  cm. Therefore, the duct diameter will be  $r = 2 + \sqrt{8} \simeq 4.83$  cm.

### Ex. 3 (A group): About differentiability

Let  $f$  be a function from an interval  $[a, b]$  to  $\mathbb{R}$  (with  $a, b \in \mathbb{R}, a < b$ ), and  $x_0 \in [a, b]$ .

1. Recall the *formal* definition of  $f$  being differentiable at this point  $x_0$  (as a pure formula with variables, well introduced with quantifiers  $\forall$  and  $\exists$ ).
2. If  $f$  is differentiable at  $x_0$ , does this mean  $f$  is continuous at  $x_0$ ? If yes, (try to) prove it. If not, (try to) give a counter-example.

#### Solution:

1.  $f$  is differentiable at  $x_0$  **iff**  $\exists l \in \mathbb{R}, (\forall \varepsilon > 0, (\exists \eta_{\varepsilon, x_0} > 0, (\forall y \in [a, b] \setminus \{x_0\}, (|x_0 - y| < \eta_{\varepsilon, x_0} \stackrel{\text{def}}{=} >$   
 $\left| \frac{f(x_0) - f(y)}{x_0 - y} - l \right| < \varepsilon)))$ ),
2. Yes, by multiplying by  $x_0 - y$  the previous inequality, we have  $f(y) \xrightarrow{y \rightarrow x_0} f(x_0)$ .