

Add a direct link to the Moodle page.

Consigns You should be extra cautious about small mistakes, try to redact as well as possible, and do not forget to justify every statements and every steps in your answers.

Source http://mathschallenge.net/archive/mathschallenge_3_star.pdf.

1 Absolute Sums

We recall that for all real numbers, the absolute value of x is written $|x|$, and is defined like $|x| \stackrel{\text{def}}{=} \max(x, -x) = x$ if $x \geq 0$ else $-x$.

1. Given that x and y are integers, how many different solutions does the equation $|x| + 2|y| = 100$ have?
 2. If k is a positive integer (odd or even), try to investigate the number of solutions of the equation $|x| + 2|y| = k$.
 3. What about the equation $|x| + 3|y| = k$?
-

2 Factorial Divisibility

Given that n is a positive integer, prove that $(2^n)!$ is divisible by 2^{2n-1} .

3 3 prime numbers?

Given that $(a, b, c) \in \mathbb{N}^*$ are three non-negative integers, if we have $abc + ab + bc + ca + a + b + c = 1001$, what is the value of $a + b + c$?

Hint: $1001 = 7 \times 11 \times 13$.

4 Cables

A telephone company places round cables in round ducts.

Assuming the diameter of a cable is 2 cm, what would the diameter of the duct be for two cables, three cables and four cables?

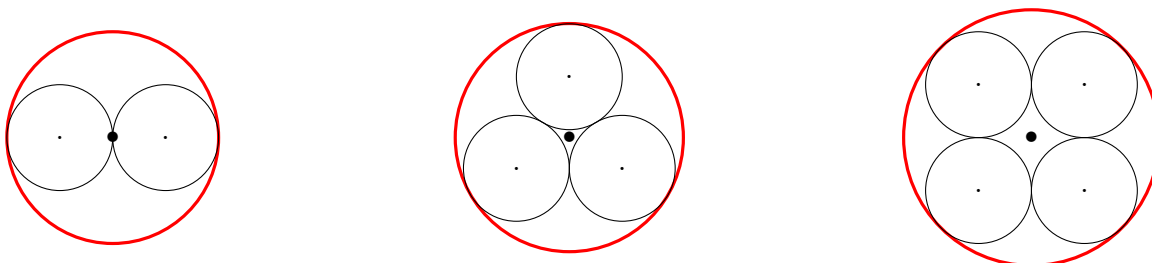


Figure 1: Cables

5 Alternating Sign Square Sum

We recall that T_n , the n^{th} triangle number is defined as $T_n \stackrel{\text{def}}{=} 1 + 2 + \dots + n$.

It can be seen that $6^2 - 5^2 + 4^2 - 3^2 + 2^2 - 1^2 = 21 = T_6$, the sixth triangle number.

Prove that for any $n \in \mathbb{N}^*$, $T_n = n^2 - (n-1)^2 + (n-2)^2 - \dots - 1^2$.

6 About continuity

Let f be a function from an interval $[a, b]$ to \mathbb{R} (with $(a, b) \in \mathbb{R}^2, a \leq b$), and $x_0 \in [a, b]$.

1. Recall the formal definition of f being continuous in this point x_0 (as a formula with variables well introduced with quantifiers \forall and \exists).
 2. If f is continuous in x_0 , does this means f is differentiable in x_0 ? If yes, try to prove it. If not, try to give a counter-example¹.
-

7 Factorial and Power of 2

Given that a , b , and c are positive integers, solve the following equation :

$$a!b! = a! + b! + 2^c.$$

8 Falling Sound

A boy drops a stone down a well and hears the splash from the bottom after three seconds (3s). Given that sound travels at a constant speed of 300 m/s and the acceleration of the stone due to gravity is 10 m/s², how deep is the well?

9 Converging Root

Consider the iterative formula, $u_{n+1} = \sqrt{u_n} + 2, \forall n \in \mathbb{N}$. By investigating the behavior it should become clear to you that, for all positive starting values, this sequence $(u_n)_{n \in \mathbb{N}}$ converges to the limit value of 4.

What form must the positive integer, m , take, for the iterative formula $u_{n+1} = \sqrt{u_n} + m, \forall n \in \mathbb{N}$, to converge to an **integral root**?

10 Bag of Balls

Three balls are placed inside a cone such that each ball is in contact with the edge of the cone and the next ball.

¹One could think of the absolute value $|\cdot| : \mathbb{R} \longleftrightarrow \mathbb{R}, x \mapsto |x|$.

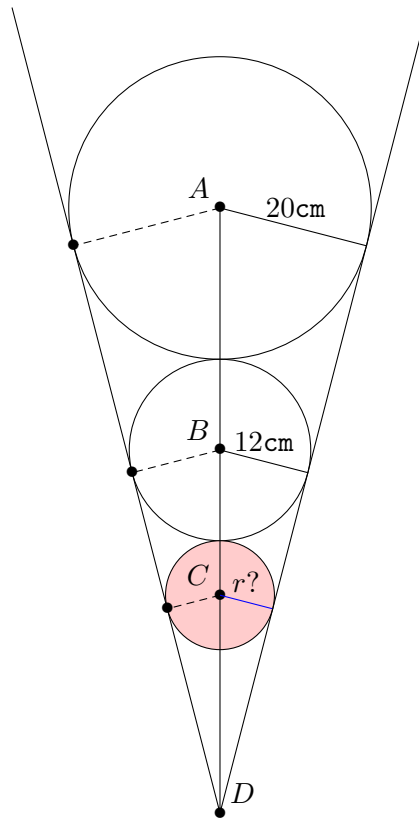


Figure 2: Bag of Balls

If the radii of the balls are 20 cm, 12 cm and r cm respectively, what is the value of r ?

11 About differentiability

Let f be a function from an interval $[a, b]$ to \mathbb{R} (with $(a, b) \in \mathbb{R}^2, a \leq b$), and $x_0 \in [a, b]$.

1. Recall the formal definition of f being differentiable in this point x_0 (as a formula with variables well introduced with quantifiers \forall and \exists).
 2. 5] If f is differentiable in x_0 , does this means f is continuous in x_0 ? If yes, try to prove it. If not, try to give a counter-example.
-

12 Corner Circle

A unit circle is placed against a right angle.

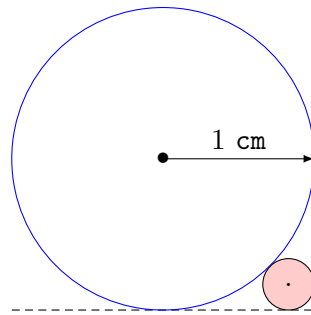


Figure 3: Corner Circle

What is radius of the smaller circle?

13 Cubes and Multiples of 7

Prove that for any number that is not a multiple of seven, then its cube will be one more or one less than a multiple of 7.

Formally, this would be written as :

$$\forall n \in \mathbb{N}^*, 7 \nmid n \stackrel{\text{def}}{\implies} 7 \mid n^3 \pm 1.$$

14 Continuity and uniform continuity

Let f be a function from an interval $[a, b]$ to \mathbb{R} (with $(a, b) \in \mathbb{R}^2, a \leq b$), and $x_0 \in [a, b]$.

1. Let call φ_1 this first formula :

$$\forall x \in [a, b], \forall \varepsilon > 0, \exists \eta_{x,\varepsilon} > 0, (\forall y \in [a, b], |y - x| < \eta_{x,\varepsilon} \stackrel{\text{def}}{\implies} |f(y) - f(x)| < \varepsilon).$$

2. Let call φ_2 this second formula :

$$\forall \varepsilon > 0, \exists \eta_\varepsilon > 0, (\forall (x, y) \in [a, b]^2, |y - x| < \eta_\varepsilon \stackrel{\text{def}}{\implies} |f(y) - f(x)| < \varepsilon).$$

3. Which formula is the strongest? Do we have $\varphi_1 \stackrel{\text{def}}{\implies} \varphi_2$ or $\varphi_2 \stackrel{\text{def}}{\implies} \varphi_1$ (or neither of these two logical implications)?

4. (Bonus:) Can you give the name of the property formally expressed by the formula φ_1 ?

5. (Bonus) And what about the second property φ_2 ?