

Remarks on grading

Prof. Jai Prakash did A2 and B2, Prof. Arya K. Bhattacharya did A3 and B4, Prof. Vijaysekhar Chellaboina did A4, Lilian Besson did A1 and B1, and Prof. Satyanarayana Chirala did B3.

Marking scheme

- For these multiple choice questions, no justification was asked.
- Every question carried 1 mark, and **only one choice (A, B, C or D)** was correct.
- There was a bonus with **+2** marks, so the best possible mark is indeed **7/5!**

Differences between A and B group? There were some changes for the coefficients on the Questions Q1, Q3, Q4 and Q5. The order of the answers also changes.

Remark Just after the quiz, the solution was discussed in class by Profs. Lilian Besson for A batch, and Vijaysekhar Chellaboina for B batch. But this complete solution is also there to help you. Please come to us and ask **any** question before the Final Exam: we all are here to help you!

Solutions, with justifications

This solution is based on the version 1 given to half of the students. Solution for the version 2 will be done later.

Is the real sequence $u = (u_n)_{n \in \mathbb{N}}$ defined by $u_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n$ convergent?

- **Q.1) (Real sequence)** No, $u_n \xrightarrow{n \rightarrow +\infty} +\infty$.

A) is the good answer, because $\frac{1 + \sqrt{5}}{2} > 1$ so $\left(\frac{1 + \sqrt{5}}{2} \right)^n \xrightarrow{n \rightarrow +\infty} +\infty$. We conclude by the “product by a constant” rule: $\frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n \xrightarrow{n \rightarrow +\infty} +\infty$ because $\frac{1}{\sqrt{5}} > 0$.

Therefore, the 3 other answers are clearly false (u_n is well defined, does not converge to $\frac{1}{\sqrt{5}}$, and does not diverges to $-\infty$).

Let S be the series sum of the terms $a_n = \frac{1}{n(n+3)}$ for $n > 0$, ie $S = \sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} \frac{1}{n(n+3)}$. Is S convergent?

- **Q.2) (Real series)** Note: This question was on the first mid term exam!

Yes, this series is convergent. We have the inequality $0 < a_n < \frac{1}{n^2}$, and we know that $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges. So thanks to the comparison test, the sum of a_n also converges. The good answer is **D**).

Note: For your information, that other sum $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ is equal to $\frac{\pi^2}{6}$.

The other answers are clearly false.

A series of positive terms is obviously not always convergent. For example, the sums of $a_n = 1$ or $\frac{1}{n}$ both diverge to $+\infty$.

A series of positive terms, even with $a_n \rightarrow 0$ when $n \rightarrow +\infty$, does not necessarily converge ($a_n = \frac{1}{n}$ again is a good counter-example).

Finally, having $a_n > 0$ is not enough to say that the series will diverge to $+\infty$.

Bonus: (for +2 marks) We can quickly compute the value of the sum of the series. More details on that web page <http://goo.gl/UNdKQ0>.

$$\text{Let } N > 0, \text{ and } S_N = \sum_{n=1}^N \frac{1}{n(n+3)} = \sum_{n=1}^N \left(\frac{1}{3n} - \frac{1}{3(n+3)} \right).$$

$$\text{So } S_N = \frac{1}{3} \left(\sum_{n=1}^N \frac{1}{n} \right) - \frac{1}{3} \left(\sum_{n=1}^N \frac{1}{n+3} \right) \xrightarrow{N \rightarrow +\infty} \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} \right) = \frac{11}{18}.$$

$$\text{Note: We check that } \sum_{n=1}^{+\infty} \frac{1}{n(n+3)} = \frac{11}{18} < \frac{\pi^2}{6} = \sum_{n=1}^{+\infty} \frac{1}{n^2}.$$

What are the solution(s) of this homogeneous 1st order differential equation $y' + 2y = 0$?

- **Q.3) (1st order differential equation)** To find the solutions of this 1st order differential equation $y' + 2y = 0$, we apply the method seen in class.

First, the domain of the equation is \mathbb{R} .

We solve the homogeneous case, and find that the solutions are the functions y of the form $y(x) = Ce^{-2x}$ (for one constant $C \in \mathbb{R}$).

What are the solution(s) of this homogeneous 1st order differential equation $y' + 4y = 0$?

• **Q.3')** (1st order differential equation) To find the solutions of this 1st order differential equation $y' + 4y = 0$, we apply the method seen in class.

First, the domain of the equation is \mathbb{R} .

We solve the homogeneous case, and find that the solutions are the functions y of the form $y(x) = Ce^{-4x}$ (for one constant $C \in \mathbb{R}$).

And for this equation $y' + 2y = x^2$, with the initial value problem $y(1) = 2$, which of these functions is a valid solution:

• **Q.4)** (1st order differential equation with initial value) We just have to check. More details are available on-line at <http://goo.gl/12bmm8>. Only the answer **B**) was possible:

For **A**), the proposed solution $y(x) = \frac{1}{2}x^2 - \frac{1}{2}x + 2$ is not even a solution of the differential equation. It satisfies $y(1) = 2$, but not $y' + 2y = x^2$.

For **B**), the proposed solution $y(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4} - \frac{1}{4}e^{2-2x}$ is the valid one.

For **C**), the proposed solution $y(x) = \frac{1}{2}x^2 + 4e^{-2x}$ is not a valid solution. And it does not satisfy $y(1) = 2$.

For **D**), the proposed solution $y(x) = \frac{1}{4} + \frac{7}{4}e^{-2x}$ is not a valid solution. And it does not satisfy $y(1) = 2$.

And for this equation $y' + 4y = -x^2$, with the initial value problem $y(1) = 0$, which of these functions is a valid solution:

• **Q.4')** (1st order differential equation with initial value) We just have to check. More details are available on-line at <http://goo.gl/4YehEg>. Only the answer **B**) was possible:

For **A**), the proposed solution $y(x) = -\frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{32} + \frac{5}{32}e^{4-4x}$ is the valid one.

For **B**), the answer “there is no solution for this initial value problem!” is wrong, because **A**) is a solution of this IVP (initial value problem).

For **C**), the proposed solution $y(x) = -\frac{1}{2}x^2 + \frac{1}{4}x - \frac{1}{16} + \frac{5}{16}e^{4(x-1)}$ is not a valid solution of the differential equation. And it does not satisfy $y(1) = 0$.

For **D**), the proposed solution $y(x) = 2e^{-4x}$ is not a valid solution of the differential equation. And it does not satisfy $y(1) = 0$.

What are the solution(s) of this non-homogeneous 2nd order differential equation $y'' + 2y' - 8y = 0$?

- **Q.5)** We apply the method explained in class. The domain is \mathbb{R} .

For the homogeneous part, the characteristic equation $r^2 + 2r - 8 = 0$, which has -4 and 2 as distinct real solutions. So $y(x) = C_1e^{-4x} + C_2e^{2x}$ are the solutions of the homogeneous equation (with $C_1, C_2 \in \mathbb{R}$ two constants).

What are the solution(s) of this non-homogeneous 2nd order differential equation $y'' - 2y' - 8y = 0$?

- **Q.5')** We apply the method explained in class. The domain is \mathbb{R} .

For the homogeneous part, the characteristic equation $r^2 - 2r - 8 = 0$, which has 4 and -2 as distinct real solutions. So $y(x) = C_1e^{4x} + C_2e^{-2x}$ are the solutions of the homogeneous equation (with $C_1, C_2 \in \mathbb{R}$ two constants).