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Roll Number: 10⁷ f

Section: B17

7/5

Note: Every question carries 1 mark. (You can use the back of the paper for rough work.)

As last time, no need to justify your choice, and for a multiple choice question only one (A, B, C or D) is correct.

- Q.1) What are the solution(s) of this homogeneous 2nd order differential equation $y'' - 2y' - 8y = 0$?

1 $y(x) = \underline{C_1 \cdot e^{4x} + C_2 \cdot e^{-2x}}$ for any constants $C_1, C_2 \in \mathbb{R}$.
 $(x \in \mathbb{R})$

- Q.2) For this non-homogeneous equation $y' + 4y = -x^2$, we ask the initial value $y(1) = 0$. Which one of these functions is a solution?

2 A) $y(x) = -\frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{32} + \frac{5}{32}e^{4-4x}$ ✓ C) $y(x) = -\frac{1}{2}x^2 + \frac{1}{4}x - \frac{1}{16} + \frac{5}{16}e^{4(x-1)}$
 B) there is no solution for this initial value problem! D) $y(x) = 2e^{-4x}$
 → We just check that only this $y(x)$ satisfies both y'

- Q.3) Let S be the series sum of the terms $a_n = \frac{1}{n(n+3)}$ for $n > 0$, ie $S = \sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} \frac{1}{n(n+3)}$. Is S convergent?

2 A) No (S will be infinite = $+\infty$ because $a_n > 0$) C) Yes (obvious, just because $a_n \rightarrow 0$)
 B) Yes (thanks to the comparison test, the inequality $a_n < \frac{1}{n^2}$ and because $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges) ✓ D) Yes (a series of positive terms always converges!)

Bonus: (for +2 marks) Can you quickly (but carefully) compute the value of the sum of the series?

+2 $\sum_{n=1}^{+\infty} \frac{1}{n(n+3)} = ?$ $\frac{11}{18}$ (of course for the proof).

- Q.4) Is the real sequence $u = (u_n)_{n \in \mathbb{N}}$ defined by $u_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n$ convergent?

A) No, u_n is not even correctly defined!

C) Yes, and it converges to $\frac{1}{\sqrt{5}}$

B) No, $u_n \rightarrow -\infty$

D) No, $u_n \rightarrow +\infty$ ✓

Because $\frac{1}{\sqrt{5}} > 0$, and $\left(\frac{1+\sqrt{5}}{2} \right) > 1$, so $\left(\frac{1+\sqrt{5}}{2} \right)^n \rightarrow +\infty$.

- Q.5) What are the solution(s) of this homogeneous 1st order differential equation $y' + 4y = 0$?

1 Solutions are y defined on \mathbb{R} by $y(x) = C \cdot e^{-4x}$ for any constant $C \in \mathbb{R}$.

Name: Superman

Roll Number: 00π

Section: C4

7/5

Note: Every question carries 1 mark. (You can use the back of the paper for rough work.)
As last time, no need to justify your choice, and for a multiple choice question only one (A, B, C or D) is correct.

• Q.1) Is the real sequence $u = (u_n)_{n \in \mathbb{N}}$ defined by $u_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n$ convergent?

A) No, $u_n \rightarrow +\infty$ ✓

C) No, u_n is not even correctly defined!

B) Yes, and it converges to $\frac{1}{\sqrt{5}}$

D) No, $u_n \rightarrow -\infty$

Because $\frac{1}{\sqrt{5}} > 0$, and $\frac{1 + \sqrt{5}}{2} > 1$ so $\left(\frac{1 + \sqrt{5}}{2} \right)^n \rightarrow +\infty$.

• Q.2) Let S be the series sum of the terms $a_n = \frac{1}{n(n+3)}$ for $n > 0$, ie $S = \sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} \frac{1}{n(n+3)}$. Is S convergent?

A) Yes (a series of positive terms always converges!)

D) Yes (thanks to the comparison test, the inequality $a_n < \frac{1}{n^2}$ and because $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges) ✓

B) Yes (obvious, just because $a_n \rightarrow 0$)

C) No (S will be infinite = $+\infty$ because $a_n > 0$)

$\forall n \geq 1, 0 \leq a_n \leq \frac{1}{n^2}$, and $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges, so $\sum_{n=1}^{+\infty} a_n = S$ converges

Bonus: (for +2 marks) Can you quickly (but carefully) compute the value of the sum of the series?

$\sum_{n=1}^{+\infty} \frac{1}{n(n+3)} = ?$ For $n \geq 1, \frac{1}{n(n+3)} = \frac{1}{3} \left(\frac{1}{n} - \frac{1}{n+3} \right)$.

So for $N \geq 1, \sum_{n=1}^N \frac{1}{n(n+3)} = \frac{1}{3} \sum_{n=1}^N \frac{1}{n} - \frac{1}{3} \sum_{n=1}^N \frac{1}{n+3} = \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} \right) - \frac{1}{3} \left(\frac{1}{N+1} + \frac{1}{N+2} + \frac{1}{N+3} \right)$

$\xrightarrow{N \rightarrow +\infty} \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} \right) = \frac{1}{3} \cdot \frac{11}{6} = \frac{11}{18} = \sum_{n=1}^{+\infty} \frac{1}{n(n+3)}$

• Q.3) What are the solutions of this 1st order differential equation $y' + 2y = 0$?

Solutions are $y(x) = C \cdot e^{-2x}$ for any constant $C \in \mathbb{R}$ (defined for $x \in \mathbb{R}$) ✓

• Q.4) For this non-homogeneous equation $y' + 2y = x^2$, we add the initial value $y(1) = 0$. Which one of these functions is a solution?

A) $y(x) = \frac{1}{2}x^2 - \frac{1}{2}x + 2$

C) $y(x) = \frac{1}{2}x^2 + 4e^{-2x}$

B) $y(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4} - \frac{1}{4}e^{2-2x}$ ✓

D) $y(x) = \frac{1}{4} + \frac{7}{4}e^2 e^{-2x}$ only

↳ We just check that this $y(x)$ satisfies $y(1) = 0$ and $y'(x) + 2y(x) = x^2 \forall x \in \mathbb{R}$.

• Q.5) What are the solution(s) of this homogeneous 2nd order differential equation $y'' + 2y' - 8y = 0$?

We solve $r^2 + 2r - 8 = 0 \Leftrightarrow r = -4$ or $r = 2$: 2 real and distinct roots

so the solutions are

$y(x) = C_1 \cdot e^{-4x} + C_2 \cdot e^{2x}$ for any $C_1, C_2 \in \mathbb{R}$ constants. ✓