

Remarks on grading

Prof. Jai Prakash corrected the papers for the A group, and Prof. Lilian Besson for the B group.

Marking scheme

- For these multiple choice questions, no justification was asked.
- Every question carried 1 mark, and **only one choice (A, B, C or D)** was correct.
- There was a bonus with **+0.5** mark, so the best possible mark is indeed **5.5/5!**

Differences between A and B group? No difference, only the order of the questions and of the answers.

Solutions, with justifications

This solution is based on the order given to the A group.

Choice:	A	B	C	D
Q.1:	False	<i>True</i>	False	False
Q.2:	False	False	<i>True</i>	False
Q.3:	<i>True</i>	False	False	False
Q.4:	False	<i>True</i>	False	False
Q.5:	False	False	False	<i>True</i>

- **Q.1) (Limits of usual functions)** If $\beta > 0$, what is the value of this limit: $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^\beta}$?
- A) that limit does not even exist! **B)** $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^\beta} = -\infty$ C) $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^\beta} = 1$ **D)** $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^\beta} = 0$.

The correct answer here is **B**).

Let $\beta > 0$. For $x \rightarrow 0^+$, $\ln(x) \rightarrow -\infty$ on one hand; and $x^\beta \rightarrow 0^+$, so $\frac{1}{x^\beta} \rightarrow +\infty$ on the other hand.

So by the product rule, $\frac{\ln(x)}{x^\beta} \rightarrow (-\infty)(+\infty) = -\infty$ as wanted¹.

Remark 1. This problem was in the tutorial sheet on Usual Functions, problem I.5, only with α instead of β ...

Clearly, the other choices **A**), **C**) and **D**) are false if **B**) is correct.

Remark 2. Apparently, there was a small risk of confusion here. The answer **A**) might be true, depending on how we defined the existence of a limit.

We say that a limit $\lim_{x \rightarrow x_0} f(x)$, at a finite point x_0 or $\pm\infty$, exists if:

- there is a $L \in \mathbb{R}$ such that $f(x)$ converges to L at x_0 or $\pm\infty$,
- or $f(x)$ diverges to $+\infty$ or $-\infty$.

¹There is no trick here, just simply computing the limit as a product.

You can remember that there is three cases:

- finite existing limit ($L \in \mathbb{R}$, e.g. $\lim_{x \rightarrow 0} \cos(x) = 1$),
- infinite existing limit ($L = \pm\infty$, e.g. $\lim_{x \rightarrow 0} \ln(x) = -\infty$),
- non-existing limit (no L , e.g. $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$ does not exists).

• **Q.2) (Power functions)** For x one real number such that $x > 1$, this expression $x^{\frac{\ln(\ln(x))}{\ln(x)}}$ can be simplified:

- A)** $x^{\frac{\ln(\ln(x))}{\ln(x)}} = \exp(x)$ **B)** $x^{\frac{\ln(\ln(x))}{\ln(x)}} = x$ **C)** $x^{\frac{\ln(\ln(x))}{\ln(x)}} = \ln(x)$ **D)** $x^{\frac{\ln(\ln(x))}{\ln(x)}}$ is not defined!

The correct answer here is **C**).

We remember that $a^b = \exp(b \ln(a))$ for $a, b > 0$. Let $x > 1$, so we have $\ln(x) > 0$.

For $x^{\frac{\ln(\ln(x))}{\ln(x)}}$, we use twice this formula: $x^{\frac{\ln(\ln(x))}{\ln(x)}} = \exp\left(\frac{\ln(\ln(x))}{\ln(x)} \ln(x)\right) = \exp(\ln(\ln(x))) = \ln(x)$.

Remark 3. This problem was in the tutorial sheet on Usual Functions, problem II.2.1), exactly like this.

Clearly, the other choices **A**), **B**) and **D**) are false if **C**) is correct (indeed, if $x > 1$, x is always different than $\exp(x)$ and $\ln(x)$).

• **Q.3) (Inverse)** Let f be a function defined on the interval $\left(\ln\left(\frac{2}{5}\right), +\infty\right)$ by $f(x) = \ln(5e^x - 2)$.

We want to find the inverse function of f : for $x \in D_f$ and $y \in \mathbb{R}$, $y = f(x)$ **if and only if** $x = f^{-1}(y)$.

- A)** $y = f(x) \iff x = f^{-1}(y) = \ln(e^y + 2) - \ln(5)$ **B)** $y = f(x) \iff x = f^{-1}(y) = \ln(5e^y - 2)$
C) $y = f(x) \iff x = f^{-1}(y) = \ln(e^y + 5) + \ln(2)$ **D)** $y = f(x) \iff x = f^{-1}(y) = 5y - 2$.

The correct answer here is **A**).

$f : x \mapsto \ln(5e^x - 2)$ is well defined on $D_f = \left(\ln\left(\frac{2}{5}\right), +\infty\right)$, because $x > \ln\left(\frac{2}{5}\right)$ iff $e^x > \frac{2}{5}$ iff $5e^x - 2 > 0$ iff $\ln(5e^x - 2)$ is well defined.

For $x \in D_f, y \in \mathbb{R}, y = f(x)$ iff $5e^x - 2 = e^y$ iff $x = \ln(e^y + 2) - \ln(5)$.

So the function f is indeed a bijection, from $\left(\ln\left(\frac{2}{5}\right), +\infty\right)$ to \mathbb{R} , of inverse $f^{-1} : x \mapsto \ln(e^x + 2) - \ln(5)$.

Remark 4. A really similar question was in the tutorial sheet on Usual Functions, problem III.1.2) with f_6 .

The other choices **B**), **C**) and **D**) are clearly false if **A**) is correct. Indeed, if $y \in \mathbb{R}$, the four expressions $\ln(e^y + 2) - \ln(5)$, $\ln(e^y + 5) + \ln(2)$, $\ln(5e^y - 2)$ and $5y - 2$ are different.

• **Q.4) (Change of variable)** Let $a > 0, x > 0$ be two constants.

If we use the change of variable $t = u/a$, then the integral $\int_0^x \frac{1}{a^2 + u^2} du$ is equal to:

A) $\frac{1}{a^2} \int_0^{ax} \frac{1}{1+t^2} dt$ **B) $\frac{1}{a} \int_0^{x/a} \frac{1}{1+t^2} dt$** C) $\frac{1}{a} \int_0^{x/a} \frac{1}{a^2+t^2} dt$ D) $a \int_0^{x/a} \frac{1}{1+t} dt$.

The correct answer here is **B**).

For u going from 0 to x , if we define t as $t = u/a$ (allowed, because $a > 0$ so $a \neq 0$), then t goes from 0 to x/a .

And with $t = u/a$, we have $\frac{dt}{du} = 1/a$, ie “ $du = a dt$ ”. But $\int_0^x \frac{1}{a^2 + u^2} du = \frac{1}{a^2} \int_0^x \frac{1}{1 + (u/a)^2} du$.

So thanks to the change of variable theorem, we can write $\frac{1}{a^2} \int_0^x \frac{1}{1 + (u/a)^2} du = \frac{1}{a^2} \int_0^{x/a} \frac{1}{1 + (t)^2} a dt$.

So indeed we have $\int_0^x \frac{1}{a^2 + u^2} du = \frac{1}{a} \int_0^{x/a} \frac{1}{1 + t^2} dt$.

Remark 5. A similar example was studied in class on Monday 27th of October when Prof. Satya during the lecture about the change of variables method. This problem is also in the tutorial sheet on integration methods, problem I.2).

The other choices **A**), **C**) and **D**) are false if **B**) is correct.

They all look similar to something that might be obtained from the initial integral $\int_0^x \frac{1}{a^2 + u^2} du$, but cannot be derived with the change of variable $t = u/a$.

Bonus: (for +0.5 mark) With this change of variable $t = u/a$, we can conclude the computation of this integral:

$$\int_0^x \frac{1}{a^2 + u^2} du = \frac{1}{a} \int_0^{x/a} \frac{1}{1 + t^2} dt.$$

But we know a primitive of $t \mapsto \frac{1}{1 + t^2}$, it is the tan inverse, arctan!

$$\text{So } \int_0^{x/a} \frac{1}{1 + t^2} dt = \left[\arctan(t) \right]_0^{x/a} = \arctan(x/a) - \arctan(0) = \arctan(x/a).$$

$$\text{Finally, we conclude that } \int_0^x \frac{1}{a^2 + u^2} du = \frac{1}{a} \arctan(x/a) = \frac{\arctan(x/a)}{a}.$$

Remark: if $x \rightarrow 0^+$, we find for the left hand-side, as expected, that this integral goes to 0 (you know that a function of the form $x \mapsto \int_0^x f(t) dt$ is continuous, and goes to 0 if $x \rightarrow 0$). And for the right hand-side, $\arctan(x/a) \xrightarrow{x \rightarrow 0} 0$.

• **Q.5) (Integration by parts)** To (try to) compute $\int_{-1}^1 x^2 \ln(x+1) dx$, can we use an integration by parts?

- A) Yes, it works, with $f(x) = x^2, g'(x) = \ln(x+1)$ [−1, 1]
 C) Yes, it works, with $f'(x) = x^2, g(x) = \ln(x+1)$
 B) No, it fails, because x^2 is not monotonic on D) No, it fails, because $x \mapsto \ln(x+1)$ is not defined at $x = -1$.

The correct answer here is **D**).

In order to compute this integral, an integration by parts might help, because it can be used to reduce (one by one) the degree of the polynomial (x^2 , then x then 1).

So we want to differentiate x^2 and integrate $\ln(x+1)$. Let $f(x) = x^2$ and $g'(x) = \ln(x+1)$ (and $g(x) = (x+1)\ln(x+1) - (x+1) + C$, for one $C \in \mathbb{R}$).

In order to use integration by parts, we need to have:

- f and g well defined on $[-1, 1]$,
- f and g continuous on $[-1, 1]$,
- f and g differentiable on $[-1, 1]$,
- and f' and g' integrable² on $[-1, 1]$.

But here, g and g' are **not defined** at $x = -1$.

So we cannot use an integration by parts, and **it fails because $x \mapsto \ln(x+1)$ is not defined at $x = -1$.**

Remark 6. A similar example was studied in class on Tuesday 28th of October when Prof. Satya during the lecture about the change of variables method.

The other choices **A**), and **B**) are clearly false if **D**) is correct.

The statement **C**) is a “joke”: monotony has nothing to do with integration by parts!

²In fact, this hypothesis is not really required, but this is the hypotheses of the theorem as you saw it in class.