

## 1 Some change of variable

Ex.1.1) Let<sup>1</sup>  $f \in \mathcal{C}^0([-a, a], \mathbb{R})$  with  $a > 0$ . With one change of variable, prove the following geometrical properties of integrals :

(i) If  $f$  is even, show that  $\int_{-a}^a f = 2 \int_0^a f$  (this is a *shortcut* for  $\int_{-a}^a f(t)dt$  and  $\int_0^a f(t)dt$ ),

(ii) If  $f$  is odd, show that  $\int_{-a}^a f = 0$ .

Ex.1.2) Let  $a > 0$ , compute  $\int_0^x \frac{1}{a^2 + u^2} du$  for all  $x > 0$ . (Remember that we know how to integrate the function  $x \mapsto \frac{1}{1 + x^2}$ .)

Ex.1.3) For  $a, b \in \mathbb{R}, a < b$ , compute  $\int_a^b (t - a)^3(b - t)^3 dt$ .

Ex.1.4) Find one primitive of the function  $f : x \mapsto \frac{\cosh(x)}{\sinh^4(x)}$  thanks to a change of variable  $u = \sinh(x)$  (of course by carefully saying on which interval you work).

Ex.1.5) (i) For an interval of the form  $\int_a^b f(x^n) \frac{dx}{x}$  (for  $0 < a < b$ ), one could remark that the change of variable  $u = x^n$  is interesting.

(ii) Compute  $\int_1^2 \frac{1}{x(1 + x^n)} dx$  for on  $n \in \mathbb{N}^*$ .

Ex.1.6) Remember the formula for  $\cos(x), \sin(x), \tan(x)$  as a function of  $\tan\left(\frac{x}{2}\right)$  (and especially the domain where they are valid). Use them to find primitives for  $x \mapsto \frac{1}{\cos(x)}$  and  $x \mapsto \frac{1}{\sin(x)}$ , maybe with the change of variable  $u = \tan\left(\frac{x}{2}\right)$ .

Ex.1.7) Compute *one* primitive of  $f : x \mapsto \frac{x}{(1 + x^2)\sqrt{1 - x^4}}$  (on the good domain).  
One could try to write  $x^2 = \cos(u)$ .

<sup>1</sup>This notation  $f \in \mathcal{C}^0(D, \mathbb{R})$  stands for a function  $f$ , from  $D$  to  $\mathbb{R}$ , being continuous.

## 2 Integration by parts, and application to compute some primitives

Ex.2.1) Compute  $\int_0^1 \ln(1+x^2)dx$ ,

Ex.2.2) Compute  $\int_0^x \arctan(t)dt$  for all  $x > 0$ ,

Ex.2.3) (LEBESGUE's Lemma) Let<sup>2</sup>  $f \in \mathcal{C}^1([a, b], \mathbb{R})$ , show that<sup>3</sup>  $\int_a^b \sin(nt)f(t)dt \xrightarrow{n \rightarrow +\infty} 0$ .

Ex.2.4) Compute *one*<sup>4</sup> primitive of  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto (3x^2 + 5x - 12) \cos(x)$ ,

Ex.2.5) Compute *one* primitive of  $f : x \mapsto \frac{1}{\sin^2(x) \cos^2(x)}$  on a domain you will choose and specify,

Ex.2.6) Compute *one* primitive of  $f : \mathbb{R}_+^* \rightarrow \mathbb{R}, x \mapsto \frac{(1-x)^2}{\sqrt{x}}$ ,

Ex.2.7) Compute *one* primitive of  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \frac{1}{(1+x^2)^2}$  by integrating by parts.

**Bonus:** One could try to prove the general case for any  $\frac{1}{(1+x^2)^n}$ , maybe by an induction  $n \in \mathbb{N}$  (**Starting from  $n = 0$  as always, remember that we agreed  $0 \in \mathbb{N}$ .**).

## 3 Other problems linking differentiation and integration

Ex.3.1) Let  $a, b \in \mathbb{R}, a < b$ . Prove that there is a constant  $C \in \mathbb{R}$  such that :

$$\forall f \in \mathcal{C}^1([a, b], \mathbb{R}), f(a) = 0 \stackrel{\text{def}}{>} \int_a^b f^2(t)dt \leq C \int_a^b (f')^2(t)dt.$$

<sup>2</sup>This notation  $f \in \mathcal{C}^1(D, \mathbb{R})$  stands for a function  $f$ , from  $D$  to  $\mathbb{R}$ , being continuous and differentiable on  $D$ , and with a first derivative also continuous.

<sup>3</sup>One could think to write  $\sin(nt)$  as  $\mathcal{I}m(e^{int})$  and then integrate by parts, by differentiate  $f$  (which is  $\mathcal{C}^1$ ) and integrate  $e^{int}$  as  $\frac{1}{int}e^{int} + C$ , if  $n \neq 0$ .

<sup>4</sup>Remember that if  $F' = f$ , and  $G = F + c$  for one constant  $c \in \mathbb{R}$ , then  $G' = f$  also!

Ex.3.2) Compute  $I = \int_a^b \frac{\sin(x)}{\sin(x) + \cos(x)} dx$  and  $J = \int_a^b \frac{\cos(x)}{\sin(x) + \cos(x)} dx$ , by assuming<sup>5</sup> that  $[a, b]$  does not contain any element being equals to  $-\frac{\pi}{4}$  modulus  $\pi$ .

Ex.3.3) Let  $f \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}_+)$ , assume that  $f''$  is bounded on  $\mathbb{R}$ . Show<sup>6</sup> that  $\forall x \in \mathbb{R}, |f'(x)| \leq \sqrt{2} \|f''\|_\infty f(x)$ .  
Notation:  $\|f''\|_\infty$  stands for the *infinity-norm*, and is defined as the maximum of  $|f''|$  on  $\mathbb{R}$ .

Ex.3.4) Find all the functions<sup>7</sup>  $f$  of  $\mathbb{R} \rightarrow \mathbb{R}$  that satisfies:

$$\forall x \in \mathbb{R}, f(x) - \int_0^x (x-t)f(t)dt = x^2.$$

Ex.3.5) Find all the functions  $f \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$  (ie, continuous function from  $\mathbb{R}$  to  $\mathbb{R}$ ) such that  $\forall x \in \mathbb{R}, \int_0^x f(t)dt = f^2(x)$ .

Ex.3.6) (*About differentiation*) Let  $f \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$ , and  $x \in \mathbb{R}$ .  
 Compute  $\lim_{h \rightarrow 0} \frac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3}$ .  
**Bonus:** Any idea of a more general result ?

## 4 Harder problem

Ex.4.1) Let  $f \in \mathcal{C}^0([a, b], \mathbb{R})$  (a continuous function from an interval  $[a, b]$  to  $\mathbb{R}$ ).

Assume that  $\forall \alpha, \beta \in [a, b], \int_\alpha^\beta f = 0$  to prove that  $f = 0$ .

Ex.4.2) (**Hard**) Let  $f, g \in \mathcal{C}_{\text{pc}}^0([a, b], \mathbb{R})$  be two function piecewise continuous (ie, continuous on every interval of a good partition of  $[a, b]$ ).

Prove that  $(\int_a^b (f+g)^2)^{\frac{1}{2}} \leq (\int_a^b f^2)^{\frac{1}{2}} + (\int_a^b g^2)^{\frac{1}{2}}$ .

<sup>5</sup>And can you justify this hypothesis ?

<sup>6</sup>By using the TAYLOR-LAGRANGE inequality for  $f$  at the second order to find a polynomial function of degree 2 on  $h$ .

<sup>7</sup>One could prove that one solution of this functional equation has to be  $\mathcal{C}^\infty$ , ie infinitely differentiable, and then that it satisfies a certain *simple* differential equation that you should solve.

Ex.4.3) (**Bonus**) Compute<sup>8</sup>  $\lim_{n \rightarrow +\infty} \left( \frac{(2n)!}{n!n^n} \right)^{\frac{1}{n}}$ .

Ex.4.4) (**Hard**) Let  $f \in \mathcal{C}^0([a, b], \mathbb{R})$ , we want to show that  $\lim_{n \rightarrow +\infty} \left( \int_a^b |f|^n \right)^{\frac{1}{n}} = \|f\|_\infty$ .

- (i) Prove existence of one  $c \in [a, b]$  such that  $|f(c)| = \|f\|_\infty$ ,
- (ii) Let  $\varepsilon > 0$ . Show that there is on neighborhood  $[\alpha, \beta]$  on  $c$  on which  $|f(x)| \geq \|f\|_\infty - \varepsilon$  for all  $x \in [\alpha, \beta]$ ,
- (iii) Then show that the sequence is lower-bounded by the sequence of general term  $(\beta - \alpha)^{\frac{1}{n}} (\|f\|_\infty - \varepsilon)$  (for  $n > 0$ ),
- (iv) Then do the same for an upper-bound, and conclude by the squeezing theorem.

Ex.4.5) (**TAYLOR-LAGRANGE's Identity**) Let<sup>9</sup>  $f \in \mathcal{C}^n([a, b], \mathbb{R}) \cap \mathcal{D}^{n+1}((a, b), \mathbb{R})$ , with  $n \in \mathbb{N}^*$ . Show<sup>10</sup> that there is one constant  $c \in (a, b)$  such that :

$$f(b) = \sum_{k=0}^n f^{(k)}(a) \frac{(b-a)^k}{k!} + f^{(n+1)}(c) \frac{(b-a)^{n+1}}{(n+1)!}.$$

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<sup>8</sup>By naming  $u_n$  the general term of that sequence to study, define an other sequence  $v = (v_n)_{n \in \mathbb{N}}$  by  $\forall n \in \mathbb{N}, v_n = \ln(u_n)$ . One could then introduce a RIEMANN sum for  $v$  (and use the basic property of logarithms).

<sup>9</sup>You should start to be familiar with these notations :  $f$  is both a function of class  $n$  of  $[a, b]$  and  $n + 1$  times differentiable on  $(a, b)$ .

<sup>10</sup>One should use ROLLE's Theorem applied to  $g : [a, b] \rightarrow \mathbb{R}, x \mapsto \sum_{k=0}^n f^{(k)}(x) \frac{(b-a)^k}{k!} + \frac{(b-a)^{n+1}}{(n+1)!} \lambda$ , with one  $\lambda$  choose to have  $g(a) = g(b)$ .