

For these problems, a and b are real numbers, satisfying $a < b$. In any case, you should be able to quickly justify the *existence* of any integrals that you write. **Every** integrals !

Ex.1) Study¹ the limit (for $n \rightarrow +\infty$) of these sequences of general term I_n , J_n , K_n and L_n (defined for any $n \in \mathbb{N}$) :

$$\begin{aligned} \text{(i)} \quad I_n &= \int_0^1 \frac{x^n}{1+x^3} dx, & \text{(iii)} \quad K_n &= \int_0^n \frac{\sin(nx)}{n^2+x^4} dx, \\ \text{(ii)} \quad J_n &= \frac{1}{n!} \int_0^1 (\arcsin(x))^n dx, & \text{(iv)} \quad L_n &= \int_0^{n^2} \frac{\sin(nx)}{n^2+x^4} dx. \end{aligned}$$

Ex.2) Compute² the following limits :

$$\begin{aligned} \text{(i)} \quad \lim_{x \rightarrow +\infty} \int_x^{x^2} \frac{\ln(t)}{1+t^4} dt, & \quad \text{(ii)} \quad \lim_{a \rightarrow 0} \int_0^1 x^3 \sqrt{1+a^3 x^3} dx. \end{aligned}$$

Ex.3) Let $f \in \mathcal{C}^0([a, b], \mathbb{R})$. Assume $\left| \int_a^b f \right| = \int_a^b |f|$. Show³ that f keeps a constant sign on $[a, b]$.

Ex.4) Let $f \in \mathcal{C}^0([a, b], \mathbb{R})$. Let F be the function defined on $[a, b]$ by $F(x) := \int_a^x f(t) dt$. First justify that F is well defined, and then that F is k -lipschitzian⁴ (for a good $k \in \mathbb{R}_+$ to find). Is F differentiable ? If yes, what can you say about F' ?

Ex.5) Let $f \in \mathcal{C}^0([0, 1], \mathbb{R})$. Study⁵ the limit of the sequence of general term $I_n = \int_0^1 f\left(\frac{x}{n}\right) dx$ (defined for any $n \in \mathbb{N}^*$). **Bonus:** And what if f is not continuous at 0, at 1 ?

Ex.6) Let $I = (I_n)_{n \in \mathbb{N}}$ defined by : $\forall n \in \mathbb{N}, I_n \stackrel{\text{def}}{=} \int_0^1 \frac{t^n}{1+t} dt$.

- (i) Justify the existence and then compute the limit $\lim_{n \rightarrow +\infty} I_n$,
- (ii) Find a *simple* recurrence relation between I_{n+1} and I_n (for any $n \in \mathbb{N}$),

¹For the second one J_n , you should find a rough upper-bound for arcsin, and for the last one L_n , one could split in half, by using the previous one.

²For the first one, one could find an upper-bound for the quantity under the integral by some quantity which no longer depend on x , and then study the limit; for the second one, one could find lower and upper bounds for $\sqrt{1+a^3 x^3}$ by some quantity without x and then compute the two integrals.

³One could consider the sign of $\int_a^b f$ and consider two cases.

⁴Remember that this means : $|F(x) - F(y)| \leq k|x - y|$ for any $x, y \in [a, b]$.

⁵One could compare that integral with a simpler one like $\int_0^1 f(0) dx$.

(iii) And now let $S = (S_n)_{n \in \mathbb{N}^*}$ defined by : $\forall n \in \mathbb{N}^*, S_n = 1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{n+1}}{n} = \sum_{k=1}^n \frac{(-1)^{k+1}}{k}$. Use I to simplify S ,

(iv) Conclude that this sequence S is converging and compute its limit.

Ex.7) Let $f \in \mathcal{C}^0([0, 1], \mathbb{R})$. Compute⁶ the limit of the sequence $I_n \stackrel{\text{def}}{=} n \int_0^1 x^n f(x) dx$ (defined for any $n \in \mathbb{N}$).

Ex.8) Let $(f_n)_{n \in \mathbb{N}}$ a sequence of continuous functions on $[a, b]$. Assume that $\lim_{n \rightarrow +\infty} \int_a^b (f_n(x))^4 dx = 0$. Show⁷ that $\lim_{n \rightarrow +\infty} \int_a^b f_n(x) dx = 0$.

Ex.9) Let $f \in \mathcal{D}^1([a, b], \mathbb{R})$. Assume that $|f'|$ is upper-bounded on $[a, b]$ by some constant $M \in \mathbb{R}_+$, and that $f(a) = f(b) = 0$. Show that $\left| \int_a^b f \right| \leq \frac{(b-a)^2}{4} M$.

Ex.10) Let $f \in \mathcal{C}^1([0, 1], \mathbb{R})$. Prove that $\lim_{n > 0, n \rightarrow +\infty} \sum_{k=0}^{n-1} \left(f\left(\frac{2k+1}{2n}\right) - f\left(\frac{k}{n}\right) \right) = \frac{f(1) - f(0)}{2}$.

Ex.11) Compute the limit $\lim_{n \rightarrow +\infty} \sum_{k=1}^n \frac{n+k}{n^2+k^2}$.

Ex.12) Prove that $\sum_{k=1}^n \frac{e^{-n/k}}{k^2} \underset{n \rightarrow +\infty}{\sim} \frac{1}{en}$.

Ex.13) Let $f, g \in \mathcal{C}^0([0, 1], \mathbb{R})$. Assume f and g to be non-negative (everywhere on $[0, 1]$), and satisfying $fg \geq 1$. Prove⁸ that $\left(\int_0^1 f \right) \times \left(\int_0^1 g \right) \geq 1$ (as always \times stands for a product).

Ex.14) Let $f \in \mathcal{C}^0([a, b], \mathbb{C})$ (warning, f takes complex values). Assume $\left| \int_a^b f \right| = \int_a^b |f|$. Show that argument of f ($\arg(f)$) is constant (where f is not zero).

⁶One could try to guess the limit by considering some examples, and then study the difference between the sequence and this limit, which can be written as a integral (cf. Ex.5)).

⁷One could start to show that $\lim_{n \rightarrow +\infty} \int_a^b (f_n(x))^2 dx = 0$.

⁸You could be inspired by our friends Augustin Louis C. and Hermann Amandus S..

Ex.15) Let $f \in \mathcal{C}^0([a, b], \mathbb{R})$ such that $\int_0^1 f(t)dt = \int_0^1 tf(t)dt = \int_0^1 t^2f(t)dt = 0$. Prove⁹ that f has *at least* three real zeros on $[0, 1]$. Can f have more than three zeros ?

Ex.16) Let $f \in \mathcal{C}^0([0, 1], \mathbb{R})$ such that $\forall n \in \mathbb{N}, \int_0^1 t^n f(t)dt = 0$. Prove that $f = 0$ (which is, the constant function always equals to 0).

One could either use the *dominated convergence theorem of RIEMANN*, or follow these ideas.

You can prove by contradiction, assume that there is $a < b$ such that $f \geq \eta > 0$ on $[a, b]$, for one η small enough (if f is always negative, change from f to $-f$!). Then find a polynomial function which is zero at a and at b , greater than 0 on (a, b) , and of absolute value strictly smaller than 1 otherwise (ie on $[0, 1] \setminus [a, b] = [0, a) \cup (b, 1]$).

By looking to the successive powers of this polynomial, multiplied by f , use the hypothesis on f to conclude.

⁹One could prove by contradiction : if f have more than 2 zeros, and then find one polynomial function P of degree greater or equal than 2, such that $P(f(x)) \geq 0, \forall x \in [a, b]$ and finally conclude that something is wrong.