

Ex.1) Prove these inequalities:

- (i) $\forall x \in \mathbb{R}_+, x - \frac{x^3}{6} \leq \sin(x) \leq x$, and $1 - \frac{x^2}{2} \leq \cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$.
- (ii) $\forall x \in \mathbb{R}, x - \frac{x^2}{2} \leq \ln(1+x) \leq x$.
- (iii) $\forall x \in (0, \frac{\pi}{2}), 2x < \sin(2x) + \tan(x)$.
- (iv) $\forall x \in (0, \frac{\pi}{2}), \tan(x) < x + \frac{x^3}{3}$.
- (v) $\forall x \in \mathbb{R}_+ \setminus \{1\}, \frac{\ln(x)}{x-1} < \frac{1}{\sqrt{x}}$.

Ex.2) Show that these two functions are infinitely differentiable (ie, of class \mathcal{C}^∞) :

$$\phi = \begin{pmatrix} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-\frac{1}{x}} & \text{otherwise} \end{cases} \end{pmatrix}, \psi = \begin{pmatrix} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & \begin{cases} 0 & \text{if } |x| \geq 1 \\ e^{-\frac{1}{1-x^2}} & \text{otherwise} \end{cases} \end{pmatrix}.$$

Ex.3) Study these sequences $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ defined by these recurrence relations, on the interesting interval (that you have to specify), and maybe with a discussion on the first terms u_0, v_0 :

- (a) $\forall n \in \mathbb{N}, u_{n+1} = \sqrt{6 + u_n}, v_{n+1} = \sqrt{6 - v_n}$.
- (b) $\forall n \in \mathbb{N}, u_{n+1} = \frac{4u_n + 2}{u_n + 5}, v_{n+1} = 1 + \frac{1}{v_n}$.
- (c) $\forall n \in \mathbb{N}, u_{n+1} = \cos(u_n), v_{n+1} = (1 - v_n)^2$.
- (d) $\forall n \in \mathbb{N}, u_{n+1} = 1 + \frac{1}{u_n^2 - 1}, v_{n+1} = \frac{v_n}{1 + v_n}$.
- (e) $\forall n \in \mathbb{N}, u_{n+1} = \frac{\sqrt{u_n}}{\sqrt{u_n} + \sqrt{1 - u_n}}$.

Ex.4) Does the sum, the product and the composition of two convex functions are still convex ? (As always, give one proof in the general case if you think it is true, or one counter-example if you think it is false.)

Ex.5) Let $n \in \mathbb{N}^*$ and $(x_i)_{i \in [1..n]}$ being n positive real numbers.

(a) Show that :

$$\frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}} \leq (x_1 \dots x_n)^{\frac{1}{n}} \leq \frac{x_1 + \dots + x_n}{n} \leq \frac{1}{\sqrt{n}} \sqrt{x_1^2 + \dots + x_n^2}$$

- (b) Study the equality cases in every of these 3 inequalities.
- (c) Usually, these two first inequalities are written “HM \leq GM \leq AM”, where “HM” stands for the hypergeometric mean ($= \frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}}$), “GM” is the geometric mean ($= (x_1 \dots x_n)^{\frac{1}{n}}$), and “AM” is the arithmetic mean ($= \frac{x_1 + \dots + x_n}{n}$).
- (d) Is there existence (and unity) of some real numbers $(x_i)_{i \in [1..n]} = (x_1, \dots, x_n)$ such that all these inequalities are all equalities ?

Ex.6) Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be a convex function of the real variable. Show that :

- (a) If f tends to 0 at $+\infty$, then f is non-positive,
 (b) If f is non-decreasing and non constant, then f tends to $+\infty$ at $+\infty$,

Bonus: Find an example of a good convex function for these two cases.

Ex.7) Let $f \in \mathcal{C}^0([0, 1], \mathbb{R})$, differentiable on $]0, 1[$, such that $f(0) = 0$ and $f(1)f'(1) < 0$. Show that f' will have at least on zero $]0, 1[$.

Ex.8) Let $f \in \mathcal{D}^1(\mathbb{R}_+, \mathbb{R})$ such that $f(0) = f'(0) = f(1) = 0$. Show that there is (at least) one point M of the curve Γ_f representing f , other than the origin $(0, 0)$, such that the tangent to the curve at M will pass by the origin (a drawing is almost necessary for this problem).

Ex.9) Let $f \in \mathcal{D}^2(\mathbb{R}, \mathbb{R})$, and assume that it have a *finite* limit at $-\infty$ and $+\infty$. Show that f'' at least one zero.

Ex.10) (*Two classic inequalities*) Let $n \in \mathbb{N}^*$, $(x_i)_{i \in [1..n]}$ and $(y_i)_{i \in [1..n]}$ be two families of n positive real numbers. Let p and q be two positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ (p and q are then said to be *conjugated*).

Show¹ the following inequality, known as HÖLDER's inequality:

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n y_i^q \right)^{\frac{1}{q}}.$$

And then use it to show MINKOWSKI's inequality, for one constant $r > 1$:

¹One could use the convexity of $x \mapsto x^p$.

$$\left(\sum_{i=1}^n (x_i + y_i)^r \right)^{\frac{1}{r}} \leq \left(\sum_{i=1}^n x_i^r \right)^{\frac{1}{r}} + \left(\sum_{i=1}^n y_i^r \right)^{\frac{1}{r}} .$$

Bonus: As always, you can try to find out when these inequalities are equalities.