

Remark: For all these differential equations, the *variable* is always x , with $x \in I$ (and with I being an interval), and the *unknown* is y , a continuous and once differentiable function, of differential being continuous on I (ie y is of class \mathcal{C}^1).

And, of course, if the equation have one term in y'' , then y is assumed to be of class \mathcal{C}^2 (twice differentiable, of second derivative being continuous).

(**Warning:** a couple of small typing mistakes were in that document, they are indicated here in red.)

1 First order linear differential equations

Ex.1.1) Solve the following differential equations (on the correct interval, or intervals):

(a) $y' + 3y = x^2 - 4x - 2$,

(b) $y' + y = \frac{1}{1 + e^x}$,

(c) $y' \cdot \sin(x) - y \cdot \cos(x) + 1 = 0$,

(d) $2x \cdot y' + y = x^n$, with one $n \in \mathbb{N}$,

(e) $|x| \cdot y' + (x - 1) \cdot y = x^2$,

(f) $x(x^2 - 1)y' + 2y = x^2$. You could try to find three real numbers a, b, c such that¹ $\frac{1}{x(x^2 - 1)} = \frac{a}{x} + \frac{b}{x - 1} + \frac{c}{x + 1}$.

(g) $(x \ln(x)) \cdot y' - y = -\frac{1}{x}(\ln(x) + 1)$.

Ex.1.2) Without trying to solve the differential equation, determine one interval in which the solution of the given problem is certain to exist².

(a) $\ln(x)y' + y = \cot(x)$, with the initial value being $y(2) = 3$.

(b) $y' + \tan(x)y = \sin(x)$, with the initial value being $y(\pi) = 0$.

Ex.1.3) (*Variation of parameters*) If P and Q are real-valued continuous functions defined on a interval I , consider (E) the following differential equation (same notation as the one used in class):

$$y' + P(x)y = Q(x), y(x_0) = y_0$$

(a) If $Q(x) = 0$ on I , we know³ that any solution of (E) will be written as $y(x) = B \exp(-A(x))$ for all $x \in I$, (where $A(x) = \int_{x_0}^x P(t)dt$ and B is some real constant).

¹This kind of simplification is called a *Partial fraction decomposition*.

²By using what has been seen in class about the existence – and uniqueness – of a solution to a linear first-order differential equation.

³It has been seen in class on Thursday the 6th of November.

- (b) But now, if Q is not identically equal to 0, assume that $y(x) = B(x) \exp(-A(x))$ (with the same $A(x)$) is a solution of (E) ($B(x)$ is now a function, and not only a constant $B \in \mathbb{R}$: the process is called the variation of the parameter, or variation of the constant). Then find what condition this function $B : x \mapsto B(x)$ has to verify for y to be a solution of (E) . Can you find the expression of $B(x)$ (without trying to compute it)?

Ex.1.4) Find the unique solution of the following differential equations (and try to justify why are they unique):

- (a) $xy' - 2y = 4x^3\sqrt{y}$, with the initial value being $y(1) = 0$. Introducing $z = \sqrt{y}$ could be useful.
- (b) $y' = 2xy^2 + (1 - 4x)y + 2x - 1$. One could remark that the function $y : x \mapsto y(x) = 1$ is a constant solution, and introducing a function z such that $y = 1 + \frac{1}{z}$ could be useful.

2 Second order linear differential equations

Ex.2.1) Solve the following differential equations (on the correct interval, or intervals):

- (a) $y'' + y' - 6y = 15x^2 - 4x - 3$,
- (b) $y'' + y' = x^4 - 8x^3 + 7x^2 - 5x$,
- (c) $y'' + 4y = 3 + 3x$,
- (d) $y'' + 3y' + 2y = e^x$,
- (e) $y'' + 3y' + 2y = e^{-x}$,
- (f) $y'' - 3y' + 2y = (-3x^2 + 10x - 7)e^x$,
- (g) $y'' + 4y' + 4y = (16x^2 + 16x - 14)e^x$,
- (h) $y'' + y' - 2y = 8 \sin(2x)$,
- (i) $y'' - 5y' + 6y = \frac{\cosh^2(x)}{e^x}$,
- (j) $y'' - 2y' + 5y = -4e^{-x} \cos(x) + 7e^{-x} \sin(x) - 4e^x \sin(2x)$,
- (k) $y'' + \omega^2 y = x \sin(2x)$, with some $\omega > 0$ being fixed.

Ex.2.2) Solve the following differential equation (on the correct interval, or intervals):

$$(1+x)^2 y'' + (1+x)y' - 2 = 0.$$

Ex.2.3) By introducing the new variable $z = xy$, solve the following differential equation:

$$xy'' + 2(x+1)y' + (x+2)y = 0.$$

(Warning: it was $2(x-1)$ in the sheet, sorry about the typo!)

Ex.2.4) Let (E) be the following DE: $ax^2y'' + bxy' + cy = 0$, where a, b, c are three real numbers such that $a > 0$ and $b > 0$.

- (a) By writing $z(t) = y(e^t)$, show that y is solution (on $\mathbb{R}_+^* =]0, +\infty[$) of the original DE (E) if and only if z is solution of another second order DE with constant coefficient that you shall specify.
- (b) Give the forme of the solutions for this DE (E) on \mathbb{R}_+^* (and on \mathbb{R}_-^*) ?
- (c) **Example:** Solve $x^2y'' - xy' + y = 0$.

Ex.2.5) Consider that differential equation $(1 - x^2)y'' - xy' + y = 0$.

- (a) Solve it on $(-1, 1)$, by using the change of variable $x = \cos(t)$. (**Warning:** it was $\sin(t)$ in the sheet, sorry about the typo!)
- (b) Then solve it on *all* the other possible intervals.
Hint: for one $k \in \mathbb{Z}$, work on the interval $I_k = (2k + 1, 2k + 3)$, on which the change of variable $x = 2k + 2 + \cos(t)$ works exactly as on $(-1, 1)$ (in the previous question).
- (c) And finally what about the global solutions ? (a global solution will be a solution on every intervals I_k for each $k \in \mathbb{Z}$, plus it has to be continuous and differentiable at every points $x_k = 2k + 1$).
- (d) At the end, how many constants (C or C_1, C_2) do you have in the expression of the general solution? Is it weird to not find two constants? Why?

3 Other problems

Ex.3.1) (A small problem on integration methods.) Compute these integrals by using the good change of variable:

- (a) $\int_a^b \sin^4(x) \cos(x) dx$, if $a, b \in \mathbb{R}$ are such that $a < b$,
- (b) $\int_0^1 \sqrt{1 + t^2} dt$.

Ex.3.2) (*Delicate*) Find all the functions f , twice differentiable, from \mathbb{R} to \mathbb{R} such that:

$$\forall x, y \in \mathbb{R}, f(x + y) + f(x - y) = 2f(x)f(y). \quad (3.1)$$

Hint: first, $f(x) = 0$ is a solution, so after we can assume f to not be zero everywhere. So $f(0) = 1$.

Then, differentiate that relation 3.1 twice, first with y being a constant and differentiate in respect to x , then x is a constant and differentiate in respect to y . You should find a simple differential equation of the second order satisfied by f .