

Ex.1) Let $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto (x^4 + 2)e^{2x}$. Justify that f is infinitely differentiable, then compute $f^{(n)}(x)$ for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Ex.2) Let $x \in \mathbb{R}$. Compute $\sum_{k=0}^n k \cos(kx)$.

Ex.3) Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ (ie f is a function of $\mathbb{R} \rightarrow \mathbb{R}$), differentiable at one point $a \in \mathbb{R}$. Show that $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a)$. Is the other side¹ also true ?

Ex.4) For some $T > 0$, does the derivate of a T -periodic function is still T -periodic ? Give a really *classic* example for which $T = 2\pi$.

Ex.5) Consider this polynomial $P(X) = X^n + aX + b$, with $n \geq 2, a, b \in \mathbb{R}$. Show that P has not more than 3 distinct real roots.

Ex.6) (*DARBOUX's Theorem*) Let $f \in \mathcal{D}^1([a, b], \mathbb{R})$ (ie $f : [a, b] \rightarrow \mathbb{R}$ continuous on $[a, b]$ and differentiable on $]a, b[$). Assume $f'(a) < 0$ and $f'(b) > 0$.

- (i) Show that there is one point $c \in]a, b[$ where $f'(c) = 0$.
(We are proving that a differentiate, even without being asked to be continuous, always satisfy the *Intermediate Value Theorem*!).
- (ii) (**Bonus:**) Find one example for f such that f' is not continuous every where on $]a, b[$.

Ex.7) *Another case for ROLLE's Theorem*

- (i) Let $f \in \mathcal{C}^0([a, +\infty[, \mathbb{R})$ (ie a continuous function from $[a, +\infty[$ to \mathbb{R} , differentiable on $]a, +\infty[$. Assume that $f(a) = \lim_{x \rightarrow +\infty} f(x)$. Show that there is one point $c \in]a, +\infty[$ such that $f'(c) = 0$.
- (ii) (**extbfBonus:**) Is it still the case if the inferior bound (“ $[a$ ”) is “ $] - \infty$ ” (ie, domain of f is $] - \infty, +\infty[= \mathbb{R}$) ? We assume $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow +\infty} f(x)$. Is there at least one $c \in] - \infty, +\infty[$ such that $f'(c) = 0$? As always, if yes, prove it, if no, find a counter example.

Ex.8) Let $\alpha \in \mathbb{R}_+$. Find the class ² of the function $f_\alpha : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto |x|^\alpha$.

¹If this limit with $f(a+h) - f(a-h)/2h$ exists, will f be differentiable at a ? As always, if yes, prove it, if no, find a counter example.

²**Remember:** A function is said to be of class \mathcal{C}^0 if continuous; of class \mathcal{C}^1 if continuous and differentiable, of derivative \mathcal{C}^0 ; and of class \mathcal{C}^{k+1} if continuous, differentiable, of derivative \mathcal{C}^k ($\forall k \in \mathbb{N}$).

- Ex.9) Let $f \in \mathcal{C}^0([a, b], \mathbb{R})$, with $b > a > 0$ being fixed, differentiable on $]a, b[$. Assume that $f(a) = f(b) = 0$. Prove the existence of one $c \in]a, b[$ such that $\frac{f(c)}{c} = f'(c)$. **Bonus:** Find one example of such f .
- Ex.10) Let $a, b, c \in \mathbb{R}$. Show that this equation (x being the unknown) $18ax^{17} + 14bx^{13} + 5cx^4 - (a + b + c) = 0$ has *at least* one solution in $]0, 1[$.
- Ex.11) Let $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, verifying $\lim_{x \rightarrow +\infty} xf'(x) = 1$. Carefully show that $\lim_{x \rightarrow +\infty} f(x) = +\infty$. Can you give one example of such f .
- Ex.12) Let $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, which satisfies $\forall x \in \mathbb{R}, f \circ f(x) = \frac{x}{3} + 4$.
- Show that we have : $\forall x \in \mathbb{R}, f(\frac{x}{3} + 4) = \frac{f(x)}{3} + 4$,
 - Then prove that f' is constant (on \mathbb{R}),
 - Conclude and find the expression for f .
- Ex.13) Let $f \in \mathcal{C}^1([0, 1], \mathbb{R})$, be such that $f(0) = 0$ and $f'(x) > 0$ for any $x \in [0, 1]$. Show that there is one $m > 0$ such that, for any $x \in [0, 1]$, $f(x) \geq mx$. Give one classic example of a trigonometric function that satisfy these conditions. Which is the biggest possible³ value for this m ?
- Ex.14) Let $f \in \mathcal{C}^1([a, b], \mathbb{R})$. Assume that f has an *infinite* number of zeros. Show the existence of at least one $c \in [a, b]$ such that $f(c) = f'(c) = 0$.
- Ex.15) Let $f, g \in \mathcal{C}^1([a, b], \mathbb{R})$, differentiable on $]a, b[$. Assume $f(a) = f(b)$ and $g(a) = g(b)$. Show that $f' + fg'$ has at least one zero on $]a, b[$.
- Ex.16) Let $f \in \mathcal{D}^1(\mathbb{R}, \mathbb{R})$ such that $\lim_{x \rightarrow +\infty} (f(x) + f'(x)) = l$ for one constant $l \in \mathbb{R}$. Show that⁴ $\lim_{x \rightarrow +\infty} f(x) = l$.
- Ex.17) Let $f \in \mathcal{C}^\infty$ such that $f(0) = 0$ et $\lim_{x \rightarrow +\infty} f(x) = 0$. Show that there is an *increasing* sequence $(x_n)_{n \in \mathbb{N}}$ such that $f^n(x_n) = 0$ for any $n \in \mathbb{N}$.

³One can think of a convexity inequality to find this m .

⁴One could consider $g \stackrel{\text{def}}{=} f \times \exp : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto f(x)e^x$.