

# Week 8 Assignment

## Section A: Homework

Different questions are of different levels of depth and intensity and will not take equal time to answer. *Give to each question its due attention.* There are no “marks” to these questions. Mark them yourself on the degree of advance in the level of your understanding (and you if feel, you may share that marking with us).

A.1. You and a friend are about to drive along a twisted stretch of road in a car whose speedometer works but whose odometer (mileage counter) is broken. To find out the length of this stretch, you record the car’s velocity at 10 sec intervals, with results as shown in the accompanying table. Estimate the length of the road using

- (a) left-endpoint values
- (b) right-endpoint values.

Time (secs)	Velocity (meters/sec)
0	0
10	22
20	7.5
30	17.5
40	15
50	22
60	17.5
70	7.5
80	11
90	17.5
100	22
110	15
120	17.5

(Sols – Pg. 295, prob. 11). Note all speed values reduced to half from that in problem (and solution) to changeover from ft/sec to m/sec and appear real (I know that is not the conversion formula).

A.2 An object is dropped straight down from a helicopter. The object falls faster and faster but its acceleration decreases over time due to air resistance. The acceleration is measured in  $\text{m}/\text{sec}^2$  and recorded every second after the drop for 5 secs, as shown

t =>	0	1	2	3	4	5
a =>	10.0	6.07	3.68	2.23	1.35	0.82

- a) Find an upper and a lower estimate of the speed when  $t=5$ .  
 b) Find an upper estimate of the distance fallen when  $t=3$ .

(Sols – Pg. 296, prob. 13). All values divided by 10/32 to map from ft/sec<sup>2</sup> to m/sec<sup>2</sup>.

A.3 Use the Max-Min inequality rule to find the upper and lower bounds for the value of:

$$\int_0^1 \frac{1}{1+x^2} dx \quad - \quad (A.3.1)$$

Next use the max-min inequality rule to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1+x^2} dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1+x^2} dx$$

Add these to arrive at an improved estimate of A.3.1.

(Sols – Pg. 311, probs. 65 & 66)

A.4 Evaluate

$$\int_0^{\pi} (1 + \cos x) dx$$

(Sols – Pg. 316, prob. 10)

A.5 Evaluate

$$\int_{\pi/2}^0 \frac{(1 + \cos 2t)}{2} dt$$

(Sols – Pg. 316, prob. 15)

A.6 Evaluate

$$\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4) dt$$

(Sols – Pg. 316, prob. 20)

A.7 Evaluate

$$\int_9^4 \frac{(1 - \sqrt{u})}{\sqrt{u}} du$$

(Sols – Pg. 317, prob. 24)

A.8 Find the derivatives, first by evaluating the integral and differentiating the result, and second by differentiating the integral directly. Cross-verify the results.

a)  $\frac{d}{dx} \int_0^{\sqrt{x}} \cos t dt$ , and

b)  $\frac{d}{dx} \int_1^{\sin x} 3t^2 dt$  (Sols – Pg. 317, prob. 27 & 28)

A.9 Show that if  $k$  is a positive constant, then the area between the  $x$ -axis and one arch of the curve  $y = \sin kx$  is  $2/k$ . (Sols – Pg. 321, prob. 61)

A.10 Integrate the piecewise continuous function in the domain  $[-2, 2]$  as given below:

$$f(x) = \begin{cases} 1, & -2 \leq x < -1 \\ 1 - x^2, & -1 \leq x < 1 \\ 2, & 1 \leq x \leq 2 \end{cases} \quad (\text{Sols – Pg. 356, prob. 15})$$

A.11 Let  $f(x)$  be a continuous function. Express

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[ f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n}{n}\right) \right]$$

as a definite integral. (Sols – Pg. 357, prob. 25)

A.12 Use the result from problem A.11 to evaluate

$$\lim_{n \rightarrow \infty} \frac{1}{n^{16}} [1^{15} + 2^{15} + 3^{15} + \dots + n^{15}]. \quad (\text{Sols – Pg. 357, prob. 26b})$$

A.13 A function  $f(x)$  is defined for all real  $x$  by the formula

$$f(x) = 3 + \int_0^x \frac{1 + \sin t}{2 + t^2} dt$$

Without attempting to evaluate this integral, find a quadratic polynomial

$$p(x) = a + bx + cx^2$$

such that  $p(0) = f(0)$ ,  $p'(0) = f'(0)$  and  $p''(0) = f''(0)$ .

A.14 Find the volume of the solid generated by revolving the region bounded by the parabola  $y = x^2$  and the line  $y = 1$  about

- the line  $y = 1$
- the line  $y = 2$ , and
- the line  $y = -1$ .

(Sols – Pg. 368, prob. 47)

## Section B: Tutorial for class workout

B.1 It may be nice if average values of integrable functions obeyed the following rules on an interval  $[a, b]$ :

- $av(f + g) = av(f) + av(g)$ ; note that  $av$  denotes average

- b.  $av(kf) = k av(f)$ , for any constant  $k$
- c.  $av(f) \leq av(g)$ , if  $f(x) \leq g(x)$  on  $[a, b]$ .

Do these rules ever hold? Demonstrate correctness of your answers. (Hint: Express all  $av(\cdot)$  terms as integrals.) (Sols – Pg. 312, prob. 74)

B.2 a) Suppose the graph of a continuous function  $f(x)$  rises steadily as  $x$  moves from left to right over an interval  $[a, b]$ . Let  $P$  be a partition of  $[a, b]$  into  $n$  subintervals of length  $\Delta x = (b - a)/n$ . Show by referring to the accompanying fig. 1 that the difference between the upper and lower sums for  $f(x)$  on this partition can be represented graphically as the area of a rectangle  $R$  whose dimensions are  $[f(b) - f(a)]$  by  $\Delta x$ . (Hint: The difference  $U - L$  is the sum of areas of rectangles whose diagonals  $Q_0Q_1, Q_1Q_2, \dots, Q_{n-1}Q_n$  lie along the curve. There is no overlapping when these rectangles are shifted horizontally onto  $R$ ).

b) Suppose that instead of being equal, the lengths  $\Delta x_k$  of the subintervals of the partition of  $[a, b]$  vary in size. Show that

$$(U - L) \leq |f(b) - f(a)| \Delta x_{\max}$$

where  $\Delta x_{\max}$  is the *norm* of  $P$ , and hence  $\lim_{\|P\| \rightarrow 0} (U - L) = 0$ . (Sols – Pg. 312, prob. 77)

B.3 Suppose that  $f(x)$  has a positive derivative for all values of  $x$  and that  $f(1) = 0$ . Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t) dt ?$$

Give reasons for your answers.

- a.  $g$  is a differentiable function of  $x$ .
- b.  $g$  is a continuous function of  $x$ .
- c. The graph of  $g$  has a horizontal tangent at  $x = 1$ .
- d.  $g$  has a local maximum at  $x = 1$ .
- e.  $g$  has a local minimum at  $x = 1$ .
- f. The graph of  $g$  has an inflexion point at  $x = 1$ .
- g. The graph of  $dg/dx$  crosses the  $x$ -axis at  $x = 1$ .

(Sols – Pg. 321, prob. 67)

B.4 a. Show that the area  $A_n$  of an  $n$ -sided regular polygon in a circle of radius  $r$  is

$$A_n = \frac{nr^2}{2} \sin \frac{2\pi}{n}.$$

(Hint: A regular polygon has equal sides. Consider each side as the base of an isosceles triangle subtended at the centre.)

- b. Find the limit of  $A_n$  as  $n \rightarrow \infty$ . Is this answer consistent with what you know about the area of a circle? (Sols – Pg. 357, prob. 27)

B.5 Let  $f$  be such that

$$x \sin(\pi x) = \int_0^{x^2} f(t) dt$$

Find  $f(4)$ .

## Section C: Additional Exercises

C.1 Suppose that  $x$  and  $y$  are related by the equation

$$x = \int_0^y \frac{1}{\sqrt{1+4t^2}} dt$$

Show that  $d^2y/dx^2$  is proportional to  $y$  and find the constant of proportionality.

(Hint: Chain rule of differentiation has to be applied at some stage).

(Sols – Pg. 354, prob. 4)

C.2 Prove that 
$$\int_0^x \left( \int_0^u f(t) dt \right) du = \int_0^x f(u)(x-u) du$$

(Hint: Express the integral on the RHS as the difference of two integrals. Then show that both sides of the equation have the same derivative with respect to  $x$ .)

(Sols – Pg. 355, prob. 8)

Figures follow.

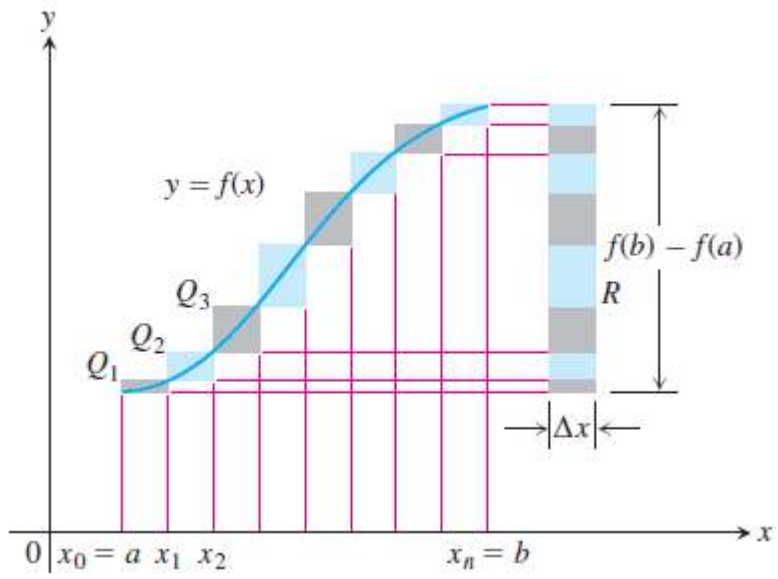


Fig. 1. Relates to problem B.2