

1 Exercises to do before tutorials.

1. Let f and g be defined by $f : x \mapsto \ln(x + \sqrt{1 + x^2})$ and $g : x \mapsto \ln(\sqrt{1 + x^2} - x)$. Determine the domains of these two functions. Find and illustrate the variations of f , and show that *when this has meaning*, we have $f(x) + g(x) = 0$. Can you give the *usual name* of this function f ?

2. Find the domain of definition, domain of continuity and domain of derivability of the following functions, and then compute their derivatives:

$$(1) f_1 : x \mapsto \ln\left(\frac{1-x}{1+x}\right),$$

$$(3) f_3 : x \mapsto \ln(\cos(x)),$$

$$(4) f_4 : x \mapsto e^{\frac{1}{1-x}},$$

$$(2) f_2 : x \mapsto \ln\left(\frac{x^2 - 4x + 4}{-x^2 - x + 6}\right),$$

$$(5) f_5 : x \mapsto \sin(x)^{\cos(x)},$$

$$(6) f_6 : x \mapsto x^{(x^x)},$$

3. Find¹ the following limit: $\lim_{x \leftrightarrow +\infty} \frac{(x^x)^x}{x^{(x^x)}}$.

4. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, compute² the next two sums: $\sum_{k=0}^n \binom{n}{k} \cosh(kx)$, $\sum_{k=0}^n \binom{n}{k} \sinh(kx)$.

5. Show that, if $\alpha > 0$, then $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^\alpha} = -\infty$.

Bonus: also prove that $\lim_{x \rightarrow 0} \ln(x) \cdot x^\alpha = 0$.

6. By the method you like, determine the following limits:

$$(a) \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x},$$

$$(e) \lim_{x \rightarrow 0} (\cos(x))^{\frac{1}{\sin^2(x)}},$$

$$(b) \lim_{x \rightarrow 0} \frac{a^x - 1}{x}, \text{ if } a > 0,$$

$$(f) \lim_{x \rightarrow 0^+} (\sin(x))^{\frac{1}{\ln(x)}},$$

$$(c) \lim_{x \rightarrow 0} \frac{(1+x)^\alpha - 1}{x}, \text{ if } \alpha \in \mathbb{R}.$$

$$(g) \lim_{x \rightarrow 0^+} x^{\sin(x)},$$

$$(d) \lim_{x \rightarrow +\infty} (\ln(x))^{\frac{1}{x}},$$

$$(h) \lim_{x \rightarrow +\infty} (e^x - 1)^{\frac{1}{x}}.$$

7. (**Hard**) For $x, y \in \mathbb{R}_+^*$, by using a simple convexity inequality with the correct function³, show

$$x \ln(x) + y \ln(y) \geq (x+y) \ln\left(\frac{x+y}{2}\right).$$

2 Exercises to do during or after tutorials.

1. Solve, on the appropriate set, the following equation: $\ln|x+2| + \ln 3 \leq \ln|3x-4|$.

¹Here, expressing powers as exponentials could be a good idea.

²Remember the expression/definition of cosh and sinh in terms of exponentials, and binomial theorem.

³Try the function $x \mapsto x \ln(x) \dots$

2. Precise the values (for x) when the following expressions have a sense, and try to simplify them as much as possible:

(1) $x^{\frac{\ln(\ln(x))}{\ln(x)}}$,

(2) $\ln_x(\ln_x(x^{(x^x)}))$ (where we remember that \ln_x is the x -based **logarithm**.)

3. Decide which of the two numbers is greater:

(a) e^π or π^e ,

(b) $2^{\sqrt{2}}$ or e ,

(c) $\ln(8)$ or 2 .

4. Compute the derivatives of the following functions, of course you will not forget to precise their domains:

(1) $f : x \mapsto \arccos(x - 1)$,

(2) $f : x \mapsto \arctan|x|$ (which is a syntactic shortcut for $\arctan(|x|)$, usually employed),

(3) $f : x \mapsto \operatorname{arcsinh}(\sqrt{x^2 - 1})$,

5. For $a, b \in \mathbb{R}$, with $a \neq b$, by using the *concavity* of $\tanh : [0, +\infty) \rightarrow \mathbb{R}, x \mapsto \tanh x = \frac{e^x - 1}{e^x + 1}$, show that

$$\frac{e^b - e^a}{b - a} < \frac{e^b + e^a}{2}.$$

6. We say that $f : [0, +\infty) \rightarrow \mathbb{R}$ is *subadditive*⁴ if, for any $x_1, x_2 \in (0, +\infty)$, f satisfies $f(x_1 + x_2) \leq f(x_1) + f(x_2)$. Prove that:

(a) if $x \mapsto \frac{f(x)}{x}$ is decreasing on $[0, +\infty)$ then f is subadditive.

(b) (**Hard**) if f is convex and subadditive⁵ on $[0, +\infty)$, then $x \mapsto \frac{f(x)}{x}$ is a decreasing function (on that interval $[0, +\infty)$).

(c) give two examples of f such that $x \mapsto \frac{f(x)}{x}$ is decreasing on $[0, +\infty)$, one with f being convex (easy), one with f not convex (harder).

7. Show that each of the equations $\sin(\cos(x)) = x$ and $\cos(\sin(x)) = x$ has *exactly* one root in $[0, \pi/2]$. Moreover, show that if x_1 and x_2 are the roots of the former and the latter equation, respectively, then $x_1 < x_2$ (**hard!**).

3 Bonus exercises.

1. Show that the following functions f_i are *bijective* (one-to-one and one-by-one), on the correct sets, and try to find an expression for their inverse (f_i^{-1}) (precise their domain):

⁴ f is said to be *additive* when $f(x_1 + x_2) = f(x_1) + f(x_2)$. One example of an additive (and therefore subadditive) function is $f : x \mapsto \alpha x$ for any constant $\alpha \in \mathbb{R}$.

⁵The initial exercise asked to work on f defined only on $(0, +\infty)$, but it was not enough to prove this part. We need to be able to compute $f(0)$, which satisfies $f(0) \leq f(0) + f(0)$, so ...

- (1) $f_1 : x \mapsto 3x - 5$,
 (2) $f_2 : x \mapsto \sqrt{3} - x$,
 (3) $f_3 : x \mapsto x^2 - 1$ (on \mathbb{R}_-),
 (4) $f_4 : x \mapsto \frac{1}{3x - 2}$,
 (5) $f_5 : x \mapsto \frac{3x + 2}{2x - 1}$,
 (6) $f_6 : x \mapsto \ln(2e^x - 1)$,

2. Same functions as the previous exercise, but now study the derivability of the functions and their inverses, then compute their derivatives. You can also try to look for some particular points where the functions and their derivatives are canceled.
3. Solve the equation $\sinh(x) = 3$, **without** using a calculator nor the formula expressing $\operatorname{arcsinh}$ as a logarithm.
4. Solve⁶ the equation $\sum_{k=0}^n \sinh(2 + kx) = 0$ for $n \in \mathbb{N}$.
5. Solve the following system:

$$\begin{cases} \operatorname{arcsinh}(x) = 2 \operatorname{arcsinh}(y) \\ 3 \ln(x) = 2 \ln(y) \end{cases}$$

6. Let $a, b \in \mathbb{R}$, solve the following equation (x being the unknown):

$$(\cosh(x) + \sinh(x))^{\operatorname{arcsinh}(x-a)} = (\cosh(x) - \sinh(x))^{\operatorname{arcsinh}(x-b)}.$$

7. Let $x, y \in \mathbb{R}$, show that, if $x = \ln\left(\tan\left(\frac{\pi}{4} + \frac{y}{2}\right)\right)$, then: $\tanh\left(\frac{x}{2}\right) = \tan\left(\frac{y}{2}\right)$, $\tanh(x) = \sin(y)$ and $\cosh(x) = \frac{1}{\cos(y)}$.

8. Simplify the following expressions:

- (i) $\arctan\left(\tan\left(\frac{5\pi}{6}\right)\right)$,
 (ii) $\arctan\left(\sin\left(\frac{\pi}{6}\right)\right)$,
 (iii) $\arctan\left(\cos\left(-\frac{\pi}{6}\right)\right)$.

9. Study these expressions (one could try to look for some trick):

- (1) $\arccos\left(\frac{1-t^2}{1+t^2}\right) + \arcsin\left(\frac{2t}{1+t^2}\right)$, for $t \in \mathbb{R}$,
 (2) $\arctan\left(\frac{x+y}{1-xy}\right) - \arctan(x) - \arctan(y)$, for $x, y \in \mathbb{R}$, with $xy \neq 1$.

⁶One could use a technique like the one used to solve $\sum_{k=0}^n \sin k\theta = 0$.

10. Solve these equations:

(1) $\tan(3 \arcsin(x)) = 1$,

(2) $\arctan(x) + \arctan(2x) = \frac{\pi}{5}$.

(3) $\arctan\left(\frac{p}{q}\right) + \arctan\left(\frac{q-p}{q+p}\right) = \frac{\pi}{5}$ with $p, q \in \mathbb{N}$ verifying $0 < p < q$.

11. Compute these sums:

(1) $\arctan(1) + \arctan(2) + \arctan(3)$,

(2) $\arctan(2) + \arctan(5) + \arctan(8)$,

12. For $n \in \mathbb{N}^*$, and $x_1, \dots, x_n \in (0, \pi)$, let $x \stackrel{\text{def}}{=} \frac{1}{n} \sum_{k=1}^n x_k$. Show that:

(a) $\prod_{k=1}^n \sin(x_k) \leq (\sin(x))^n$,

(b) $\prod_{k=1}^n \frac{\sin(x_k)}{x_k} \leq \left(\frac{\sin(x)}{x}\right)^n$.

13. For $x \in \mathbb{R}$ and $y \in \mathbb{R}_+^*$, show that $xy \leq e^x + y(\ln(y) - 1)$.

Prove also that the inequality is an equality (ie, the equality holds) if and only if $y = e^x$.