

Second half of Differential Calculus.

Remark 1. These partial solutions are only there to help, to give a hint or some element of answers.

Some problems are not corrected because they should really be easy enough for you to work on them by your own, or because they are not really exciting.

Some remarks might not be as serious as what you are used to, so deal with it.

1 A. Homework.

Ex.A.1) Let $l(t)$ be the length of the rectangle, given by $l(t) = 12 - 2t$ for $t \in [0, 6s]$. And similarly $w(t)$ is the width, by $w(t) = 5 + 2t$ for $t \in [0, 6s]$.

- The area is $A(t) = l(t)w(t)$, so $A'(t) = -2(5 + 2t) + 2(12 - 2t) = -34$.
- The perimeter is $p(t) = 2(l(t) + w(t))$, so $p'(t) = 0$: the perimeter is constant.
- The length of the diagonal of the rectangle is $d(t) = \sqrt{l^2(t) + w^2(t)}$ thanks to Pythagorean theorem. So for $t \in [0, 6s)$, d is differentiable, and $d'(t) = \frac{-2(12 - 2t) + 2(5 + 2t)}{\sqrt{l^2(t) + w^2(t)}} = -\frac{14}{\sqrt{(12 - 2t)^2 + (5 + 2t)^2}} < 0$ (nothing interesting to say about this one).

The area and the length of the diagonal are decreasing, the perimeter is constant.

Ex.A.2) Cf. figure 1. Let call θ this angle, we have $\cos(\theta) = \frac{12}{13}$, so $\theta = \arccos(12/13)$.

- (a) The top of the ladder is sliding down the wall at the velocity $\tan(\theta) * 5$ ft/sec, so at approximately 2.08 ft/sec.
- (b) The area can be expressed as $13 * \text{base}/2$, so the rate of change of the area is $33.5 \text{ft}^2/\text{sec}$.
- (c) $\theta'(t) = \frac{d}{dt}(\arccos((12 - 5t)/13)) = \frac{5}{1 - ((12 - 5t)/13)^2}$.
- (d) Nothing else to add on this real world based problem. The main interest is your ability to mathematically model a physical problem.

Ex.A.3) Here, $k > 0$, and f is defined for any $x > -1$.

What is asked is a simple application of the Taylor expansion formula for $a = 0$, $n = 1$: $f(x) \simeq L(x)$ if $x \simeq 0$, with $L(x) = f(0) + f'(0)x$, but $f(0) = 1$ and $f'(0) = k$.

- (a) $(1.0002)^{1/3} \simeq 1 + 50 * 0.0002 = 1.01$ because 0.0002 is small regarding to 1,
- (b) $(1.009)^{1/3} \simeq 1 + \frac{1}{3} * 0.009 = 1.003$ because 0.009 is small also regarding to 1.
- (c) One can verify numerically that the absolute errors made here are small regarding 1 (less than 1%).

Ex.A.4) Way out of the syllabus, do not even try to solve it.

Ex.A.5) Please understand the dx as a $\mathbb{D}elta x$, this one has *nothing* to do with the **purely formal** dx used in the notations $\frac{df}{dx}(x)$ or $\int_a^b f(x)dx$.

So, $x_0 = -1$, $\mathbb{D}elta x = 0.1$, $f(x) = 2x^2 + 4x - 3$.

- (a) The change $\mathbb{D}elta f$ is simply $f(-0.9) - f(-1) = 0.02$,
- (b) As $f'(x) = 4x + 4$, we have $f'(x_0) = 0$, the estimate $\delta f = f'(x_0)\mathbb{D}elta x$ is simply 0.
- (c) **Warning:** here again, the question paper use a notation that you might find confusing: df is a special notation, which have a **purely formal** use in the notation $f'(x) = \frac{df}{dx}(x)$, so naming this estimate df can be confusing. Just name it δf if you prefer.
- (d) The approximation error is 2%, which is *not that bad*.

Ex.A.6) Let f be defined and continuous on $[a, b] \subset \mathbb{R}$, differentiable on (a, b) . Assume that f acquires a local minimum at the point $c \in (a, b)$. At this point, we know that we can write $f'(c)$ as two different limit : the left ($h < 0$) and right ($h > 0$) limits.

(left) So $f'(c) = \lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h}$. But at the point c , f acquires a (local) minimum by hypothesis, so we know that $f(c+h) \leq f(c)$, ie $f(c+h) - f(c) \leq 0$. But when $h \rightarrow 0^-$, $h < 0$, so $f'(c) \geq 0$.

(right) So $f'(c) = \lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h}$. But at the point c , f acquires a (local) minimum by hypothesis, so we know that $f(c+h) \geq f(c)$, ie $f(c+h) - f(c) \geq 0$. But when $h \rightarrow 0^+$, $h > 0$, so $f'(c) \leq 0$.

And now we see that we have both $f'(c) \leq 0$ and $f'(c) \geq 0$, so $f'(c) = 0$. Obviously, the same can be done with a (local) maximum.

Ex.A.7) The function $f(x) = \frac{x}{x^2+1}$ is continuous and differentiable on \mathbb{R} .

$$\text{For } x \in \mathbb{R}, f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

So $f'(x) = 0$ iff $x = \pm 1$. Thanks to the first derivative test, that function f can have a local extremum only at the points $x = -1$ or $x = 1$. Graphically, or by differentiating twice, we check that $x = -1$ is a global minimum, where $f(-1) = -\frac{1}{2}$ and $x = 1$ is a global maximum, where $f(1) = \frac{1}{2}$.

Remark: one other way to prove this was to remember that $\forall a, b \in \mathbb{R}, |ab| \leq \frac{a^2+b^2}{2}$, so here with $a = x, b = 1$, we add directly that $-\frac{1}{2} \leq f(x) \leq \frac{1}{2}$. And we just check that at $x = \pm 1$, these extrema values are taken.

Ex.A.8) Let $f(x) = -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}$ if $x \leq 1$ and $f(x) = x^3 - 6x^2 + 8x$ if $x > 1$. f is continuous and differentiable on $(-\infty, 1)$ and on $(1, +\infty)$, the only point we should look more carefully is $x = 1$.

$f(1) = -\frac{1}{4}1^2 - \frac{1}{2}1 + \frac{15}{4} = 3$ and $\lim_{x \rightarrow 1^+} x^3 - 6x^2 + 8x = 1^3 - 6 \cdot 1^2 + 8 \cdot 1 = 3$. So f is continuous at $x = 1$.

On the left, $f'(x) = -\frac{1}{2}x - \frac{1}{2}$ so $f'(1) = -1$; and on the right, $f'(x) = 3x^2 - 12x + 8$ so $\lim_{x \rightarrow 1^+} f'(x) = -1$.

So f is also differentiable at $x = 1$.

Therefore f is continuous and differentiable everywhere on \mathbb{R} .

Now we can use the first derivative test, and look for the critical point. Let $x \in \mathbb{R}$.

If $x \leq 1$ $f'(x) = 0$ iff $-\frac{1}{2}x - \frac{1}{2} = 0$ iff $x = -1$.

At this point, $f''(-1) = -\frac{1}{2} < 0$, so it is a maximum.

Without any more knowledge, we cannot tell between local or global maximum, but the left part of f is a quadratic polynomial, of main coefficient being negative.

So we can conclude that at $x = -1$, the left part of f have a global maximum.

But as $\lim_{x \rightarrow +\infty} f(x) = +\infty$, we can also conclude that it is not a global maximum for f , only local.

If $x > 1$ $f'(x) = 0$ is a quadratic equation, of discriminant $\Delta = 12^2 - 4 \cdot 3 \cdot 8 = 144 - 12 \cdot 8 = 144 - 96 = 48 > 0$, so there is exactly two real solutions, $x_{1,2} = \frac{12 \pm \sqrt{48}}{6}$, with $x_1 < x_2$.

At x_1 , $f''(x) = 6x - 12$ will take the a negative value, and at x_2 , $f''(x_2)$ will be positive. So x_1 is a local maximum, and x_2 a local minimum.

Ex.A.9) Let $f : [a, b] \rightarrow \mathbb{R}$, with $a, b \in \mathbb{R}$ such that $a < b$, and assume that f is continuous on $[a, b]$, differentiable on (a, b) , with $f(a) = f(b)$. Lets show that there is a $c \in (a, b)$ such that $f'(c) = 0$.

f is continuous on $[a, b]$, so thanks to the **Extreme Value Theorem**, f is bounded and attains its bounds on $[a, b]$. Let M be the maximum of f .

- First case, if $M > f(a)$. Let c be a point such that $f(c) = M$, thanks to the second conclusion of the EVT. c is a point where f has a local maximum, and $c \neq a, b$ because $f(c) \neq f(a), f(b)$ (so $c \in (a, b)$). Therefore, the **First Derivative Theorem** gives that $f'(c) = 0$. Done!

- Second case, if $M = f(a)$. Consider now m the minimum of f .
 - If $m < f(a)$, we do the same as previously, and the EVT gives one point $c \in (a, b)$ such that $f(c) = m$, and so $f'(c) = 0$. Done!
 - In this last case, f is constant on $[a, b]$, and as $a < b$, one could define c as $\frac{a+b}{2}$ and we have $f'(c) = 0$. Done!

In every of these cases, we conclude with having a $c \in (a, b)$ satisfying $f'(c) = 0$.

Ex.A.10) We have two cases here:

- That f is well defined, continuous and differentiable on $[\frac{1}{2}, 2]$, and on this interval, $f'(x) = 1 - \frac{1}{x^2}$, and $f(\frac{1}{2}) = f(2) = \frac{5}{2}$, so the average rate $\frac{f(b) - f(a)}{b - a}$ is 0 here. Hence, the only possible c is 1 (because $1 - \frac{1}{x^2} = 1$ iff $x^2 = 1$ iff $x = 1$ because $x > 0$ anyway).
- That second f is well defined and continuous on $[1, 3]$, and differentiable on $(1, 3]$.

Ex.A.11) f is not continuous at $x = 1$.

Ex.A.12) a, b, m have to be such that f is continuous on $[0, 2]$ and differentiable on $(0, 2)$.

- Continuity at $x = 0$. We have $\lim_{x \rightarrow 0^+} f(x) = a$, so a has to be 3.
- Continuity at $x = 1$. We have $f(1) = m + b$, and $\lim_{x \rightarrow 1^-} f(x) = 2 + a$, so $m + b = 2 + a$ is another condition.
- Differentiability at $x = 0$ is not required, but at $x = 1$ it is. From the left, $f'(x) = -2x + 3$ if $0 < x < 1$, so $\lim_{x \rightarrow 1^-} f'(x) = 1$.
From the right, $f'(x) = m$ if $1 \leq x \leq 2$, so $f'(1) = m$, and so $m = 1$ is another condition.

Finally, we conclude by saying that $a = 3, m = 1, b = 2 + a - m = 4$ are the only possible values for a, b, m for f to verify the hypotheses of the mean value theorem on $[0, 2]$.

Ex.A.13) The zero of the given polynomial are (clearly) $z_0 = 0, z_1 = 9$ and $z_2 = 24$. The zero of its derivative are $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$, with $a = 3, b = -66, c = 216$, and $\Delta = b^2 - 4ac = 66^2 - 4 * 3 * 216 = 1764 = 42^2 > 0$. So $x_{1,2} = \frac{66 \pm 42}{6}$, hence $x_1 = 4, x_2 = 18$. We can see that $z_0 < x_1 < z_1 < x_2 < z_2$: the zero of y' are interlaced between the zeros of y .

Ex.A.14) This problem is the general case of the previous one. Let $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = x^n + \sum_{k=n-1}^0 a_k x^k$ and $p'(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \dots + a_1$ its derivative.

Let α, β be two consecutive zeros of p (ie, no other zero of p between them). Then we have $p(\alpha) = 0 = p(\beta)$, and p is continuous on $[\alpha, \beta]$, and differentiable on (α, β) because it is of class C^1 on \mathbb{R} . So we can apply ROLLE's Theorem to conclude that there is a $c \in (\alpha, \beta)$ such that $p'(c) = 0$.
Conclusion: so between every two zeros of p there lies a zero of p' .

Remark 2. (Out of the syllabus!) For a more general version of this result, for a complex polynomial, one could be interested to take a look at LUCAS's Theorem (also called the GAUSS-LUCAS's Theorem), for example on this Wikipédia page https://en.wikipedia.org/wiki/Gauss-Lucas_theorem.

Ex.A.15) Let $a, b \in \mathbb{R}_+^*$, with $a < b$. The function $f : [a, b] \rightarrow \mathbb{R}, x \mapsto \frac{1}{x}$ is well defined and continuous on $[a, b]$, and differentiable on (a, b) . We know $f'(x) = -\frac{1}{x^2}$, let solve $f'(c) = \frac{f(b) - f(a)}{b - a}$ iff $-\frac{1}{c^2} = \frac{\frac{1}{b} - \frac{1}{a}}{b - a} = -\frac{1}{ab}$ iff $c^2 = ab$ iff $c = \sqrt{ab}$ because $a, b > 0$.

Ex.A.16) (quickly) $f'' = g''$ iff there exists $a, b \in \mathbb{R}$ such that $\forall x \in D, f(x) = g(x) + ax + b$.

Ex.A.17) (quickly) Cf. the figure 2.

- (a) When the slope s' is positive, resp. when it is negative.
- (b) The velocity $s'(t) = 0$ for approximately $t = 2.2s, t = 6s, t = 9.5s$.
- (c) The acceleration $s''(t) = 0$ for approximately $t = 4s, 8s, 12s$ (inflection points).
- (d) Acceleration is negative for $t \in [0, 4s] \cup [8s, 12s]$ and positive for $t \in [4, 8s] \cup [12s, 15s]$.

Ex.A.18) The critical points of f (where $f'(x) = 0$) are $x = 1, 2, 4$. At these points, we just have to compute the sign of $f''(x)$ to conclude. For any $x \in \mathbb{R}$, $y''(x) = 2(x-1)(x-2)(x-4) + (x-1)^2((x-4) + (x-2))$

$x = 1$ $y''(1) = 0$, so 1 is a point of inflection.

$x = 2$ $y''(2) = -2 < 0$ so 2 is a point of (local) maximum.

$x = 4$ $y''(4) = 18 > 0$ so 4 is a point of (local) minimum.

Ex.A.19) I am from France, we do not use feet. We use, as every one else is supposed to, **meters**.
This problem is not that hard, so you can do it yourself.

Ex.A.20) Cf. figure 4.

- a) $y'(0) = d$, and graphically we clearly see that $y'(0) = 0$ as the curve has a horizontal tangent at $x = 0$.
- b) $y'(-L) = 3aL^2 - 2bL + c$, and graphically again, $y'(-L) = 0$.
- c) If $y(-L) = H$, and $y(0) = 0$, we check that the proposed cubic polynomial satisfies exactly the same 4 conditions (values of y and y' at $-L$ and 0) as y , so by uniqueness, we can indeed conclude that :

$$\forall x \in \mathbb{R}, y'(x) = H \left(2 \left(\frac{x}{L} \right)^3 + 3 \left(\frac{x}{L} \right)^2 \right).$$

Ex.A.21) To conclude.

Ex.A.22) The limit is 0, with both method.

Ex.A.23) That limit is an , for any $n > 0$.

Ex.A.24) That limit is 1.

Ex.A.25) Boring, and purely mechanical. Just compute $f(-2), f'(-2), f''(-2), f'''(-2), f^{(4)}(-2)$.

Finally, we have, for every¹ $x \in \mathbb{R}$, we have $f(x) = x^4 + x^2 + 1 = (x+2)^4 - 8(x+2)^3 + 25(x+2)^2 - 36(x+2) + 21$.

Ex.A.26) The MACLAURIN expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots = \sum_{k \geq 0} \frac{x^k}{k!}.$$

At $x = 1$, going up to the third order, this expansion gives the approximation $e \simeq 1 + \frac{1^1}{1!} + \frac{1^2}{2!} = 2.5$.

So with the approximation $e = 2.732$, the absolute error is $\left| \frac{2.732 - 2.5}{2.732} \right| \simeq 0.0803$, so an error of approximately 8%, which is not really good.

With the remainder formula, we have an upper bound of the form $M \frac{x^3}{3!}$ where M is an upper bound of e^x for $x \in (0.9, 1.1)$ (for example). So we can take $M = 2.732^1 \cdot 1 \simeq 3.02$, $x = 1$, and so the bound is approximately 0.5. Not exactly sure of what the question paper is asking here.

¹This is a special case when a finite order TAYLOR approximation is an equality everywhere, only for polynomials!

At $x = -1$, going up to the order 4, the expansion gives the approximation $\frac{1}{e} \simeq 1 + \frac{(-1)^1}{1!} + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} = 0.5 - \frac{1}{6} = 0.1667$. So with the approximation $\frac{1}{e} = 0.3678$, the absolute error is $\left| \frac{0.3678 - 0.1667}{0.3678} \right| \simeq 0.5469$, so an error of approximately 55%, which is really too much!

Ex.A.27) Should be done more in details...

- For $k > 0, x > -1$, $f : x \mapsto (1+x)^k$ is continuous and differentiable. The second order TAYLOR's formula gives $f(x) \simeq 1 + kx + k(k-1)\frac{x^2}{2}$ when $x \simeq 0$, which is the quadratic approximation of $f(x)$.
- Let $k = 3$. Numerically, we find that the error $|f(x) - (1 + kx + k(k-1)\frac{x^2}{2})|$ is less than 1% if $x < 0.01$ itself.
- Let $k = 3$. Numerically, we find that the error $|f(x) - (1 + kx + k(k-1)\frac{x^2}{2})|$ is less than 10% if $x < 0.4642$ itself.
- Remark: the last two question should be done more precisely with the Remainder Formula. The solution was done by numerically finding an approximately good threshold to have the error less than 1% or 10%.

2 B. Tutorial for class workout.

Ex.B.1) Gross!

Ex.B.2) Out of the syllabus.

Ex.B.3) See the figure, it is pretty easy anyway. Part a) was done in class.

Ex.B.4) Obvious by successive application of ROLLE's theorem.

General result is also easy.

We can use to prove that a cubic polynomial cannot have more than three real zeros, because its third derivative is a constant so do not have any zero.

Ex.B.5) If you tried carefully that problem, you might have found that the statement was not precise enough. What is a, x, f, c_2 ? Which variable is depending of the others? This can be your job, to try to improve this.

Ideally, the exercise should say² : let $f : \mathbb{R} \rightarrow \mathbb{R}$, of class \mathcal{C}^2 , and one $a \in \mathbb{R}$. For any $x \in \mathbb{R}$, there is one c_2 between a and x (or between x and a , depending on their respective position) such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c_2)}{2}(x-a)^2. \quad (2.1)$$

So this can be used as the question is asking to have a condition on the sign of f'' on an entire interval.

Assume f to be continuous, with continuous first and second derivatives, and $f'(a) = 0$.

- If $f'' \leq 0$ throughout a (small) interval $I = [\alpha, \beta]$ whose interior (α, β) contains a , then we can use the equation 2.1, with a being fixed, and **any** $x \in I$, and c_2 between a and x , so in the interior of I .

Then, we have $f(x) = f(a) + f'(a)(x-a) + \frac{f''(c_2)}{2}(x-a)^2 = f(a) + \frac{f''(c_2)}{2}(x-a)^2$. But $f''(c_2) \leq 0$, and $(x-a)^2 \geq 0$, so $f(x) \leq f(a)$ for any $x \in I$.

This is exactly saying that f has a local **maximum** at the point a .

²That statement is called the TAYLOR's theorem with an explicit formula for the remainder. If needed, take a look at that Wikipédia page: https://en.wikipedia.org/wiki/Taylor'_theorem#Explicit_formulae_for_the_remainder for examples and a proof.

- b. Exactly the same can be done with a local minimum, as if $f''(c_2) \geq 0$, then $f(x) \geq f(a)$ for any x in this small interval I .

Ex.B.6) A little bit out of the syllabus, as the Taylor expansion as seen in lectures are only for a real variable. Anyway, what could have been possible is simply use the generic form of the Taylor expansion of $\exp(x)$ valid for any $x \in \mathbb{C}$, and the ones for \cos and \sin valid on \mathbb{R} , and use the basic property of $i^2 = 1, i^4 = -1$ to conclude.

3 C. Additional Exercises.

Ex.C.1) To be concluded.

Ex.C.2) For a fixed integer $m \in \mathbb{N}$, at the point $x = 0$, the TAYLOR expansion is the binomial expansion given by the binomial theorem. For $|x| < 1$, it obviously converges as there is a finite number of terms ($m + 1$):

$$(1 + x)^m = \sum_{k=0}^m \binom{m}{k} x^k$$